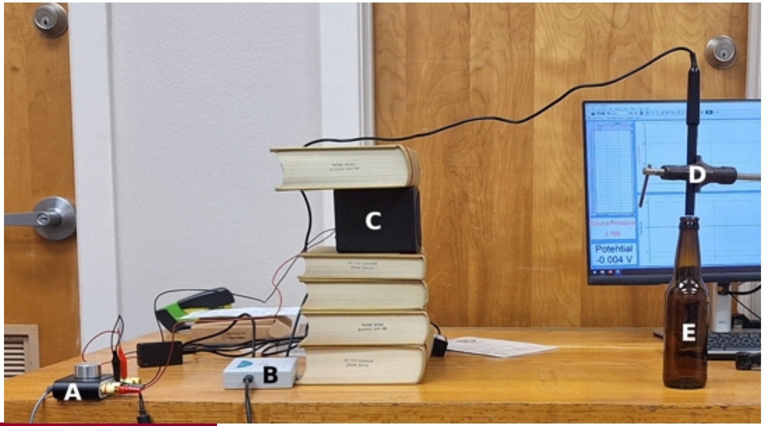
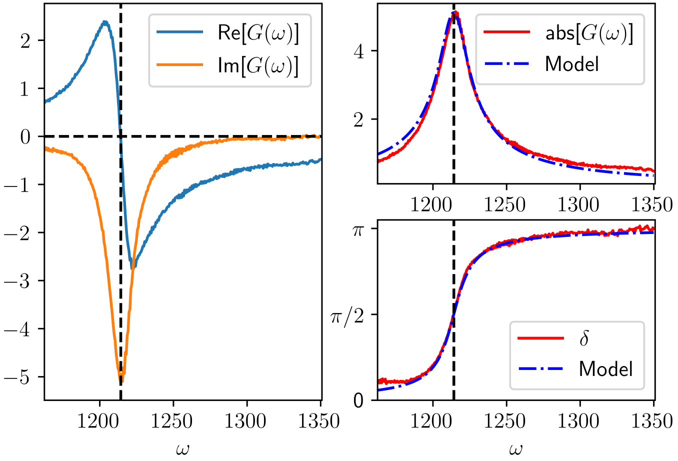
Beer bottles as acoustical resonators: a teaching tool for the damped driven oscillator model





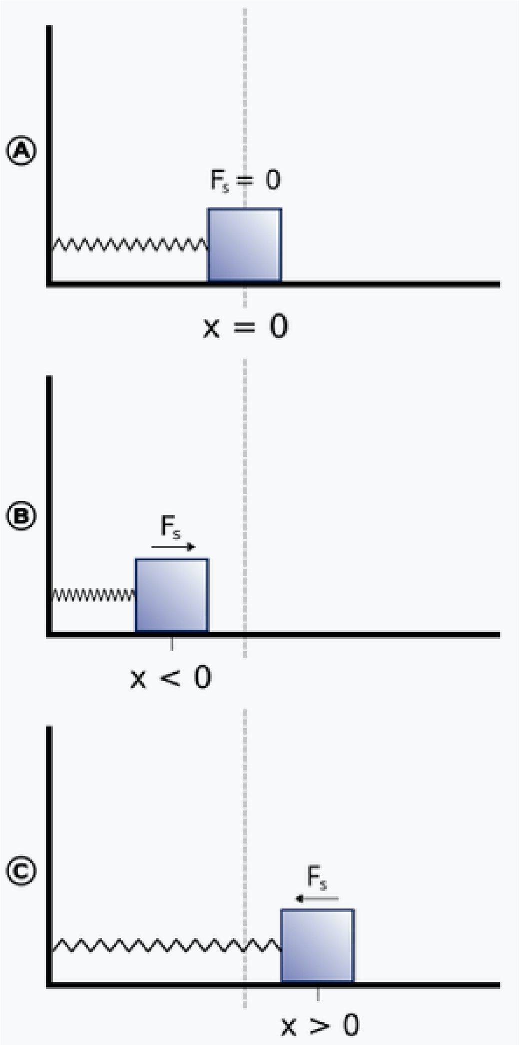
Dr. David Kordahl and Emma Foster
Department of Physics and Engineering
Centenary College of Louisiana
Louisiana Academy of Sciences – 2025 Meeting

Outline

- **Review of driven-damped oscillator model**
- **Review of Helmholtz resonator model**
- **Three approaches to extracting G(ω)**
  - Steady-state signals
  - Incoherent signals (two measurements)
  - Coherent signals (two microphones)
- **Conclusion**

A favorite toy

- The harmonic oscillator maps onto many different physical systems that oscillate



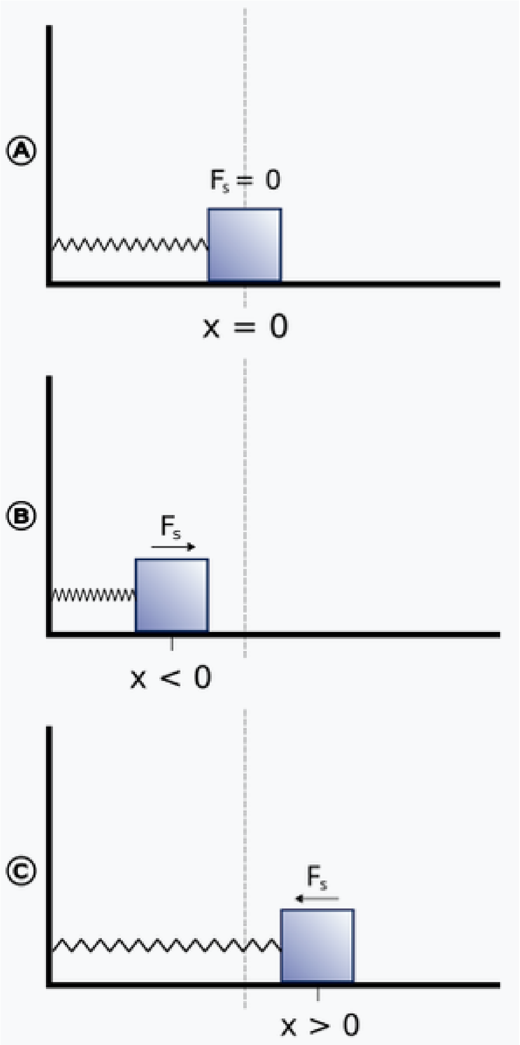
From the Wikipedia article on the HOs

A favorite toy

- **The harmonic oscillator maps onto many different physical systems that oscillate**

- **Two "tunable parameters"**
  - Block mass ***m***
  - Spring stiffness ***k***

$$\ddot{x} = -\frac{k}{m}x$$



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A favorite toy

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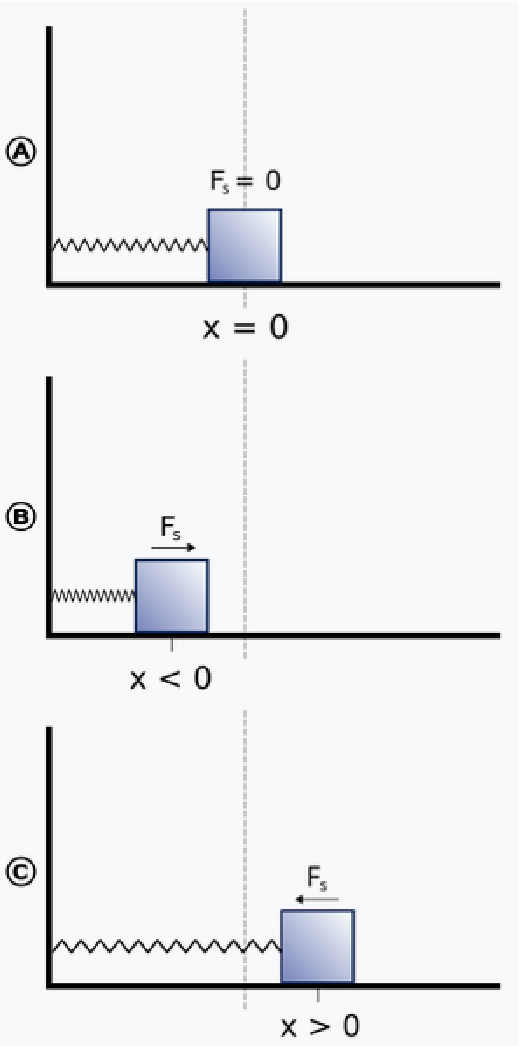
- **Two "tunable parameters"**
  - Block mass ***m***
  - Spring stiffness ***k***

$$\ddot{x} = -\frac{k}{m}x$$

- **These variables control oscillation frequency**

$$x = x_0 \cos(\omega_0 t - \delta)$$
$$\ddot{x} = \omega_0^2 x$$
$$\rightarrow \omega_0^2 = \frac{k}{m}$$



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Engineering some reality

- The obvious next step: **add a damping term**

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Engineering some reality

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- Taylor's "Classical Mechanics" gives **solution** as

$$x(t) = Ae^{-\beta t}\cos(\omega_1 t - \delta).$$

$$\omega_1 = \sqrt{\omega_o^2 - \beta^2}.$$

$$(\text{decay parameter}) = \beta$$

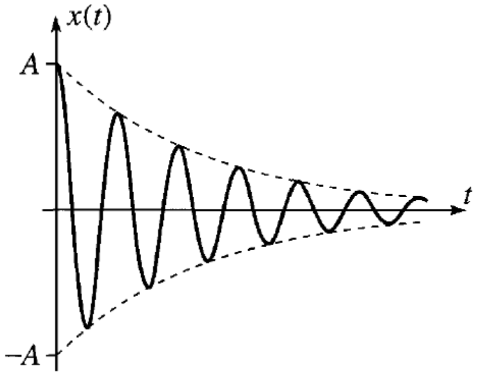

Figure 5.11 Underdamped oscillations can be thought of as simple harmonic oscillations with an exponentially decreasing amplitude $Ae^{-\beta t}$. The dashed curves are the envelopes, $\pm Ae^{-\beta t}$.

Adding external force

- One more step: **add an external force**

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = F(t)/m$$

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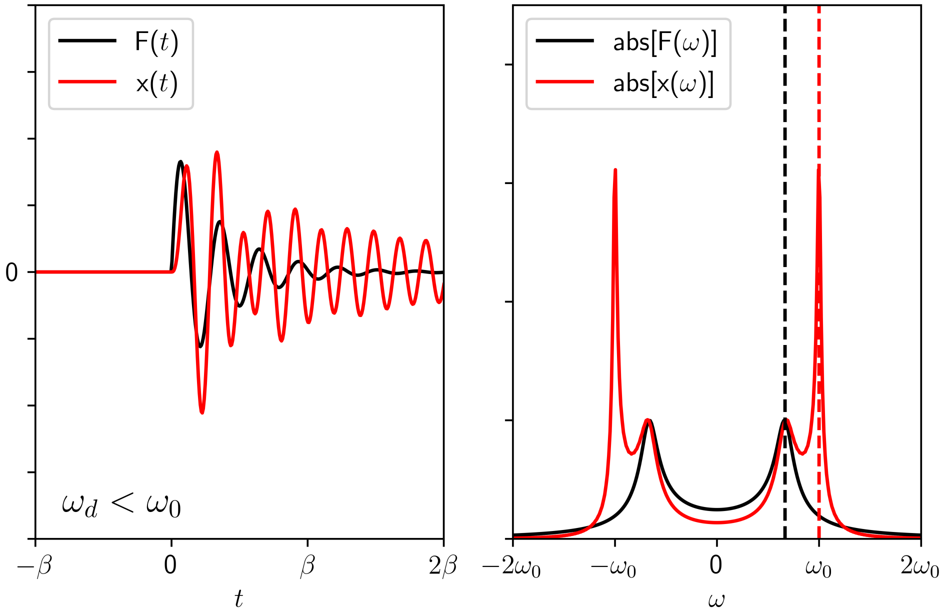

Fourier transform of force and oscillations

$$\mathscr{F}\left[f(t)\right] = \tilde{f}(\omega) = \int_{-\infty}^{+\infty} f(t)e^{-i\omega t}dt$$

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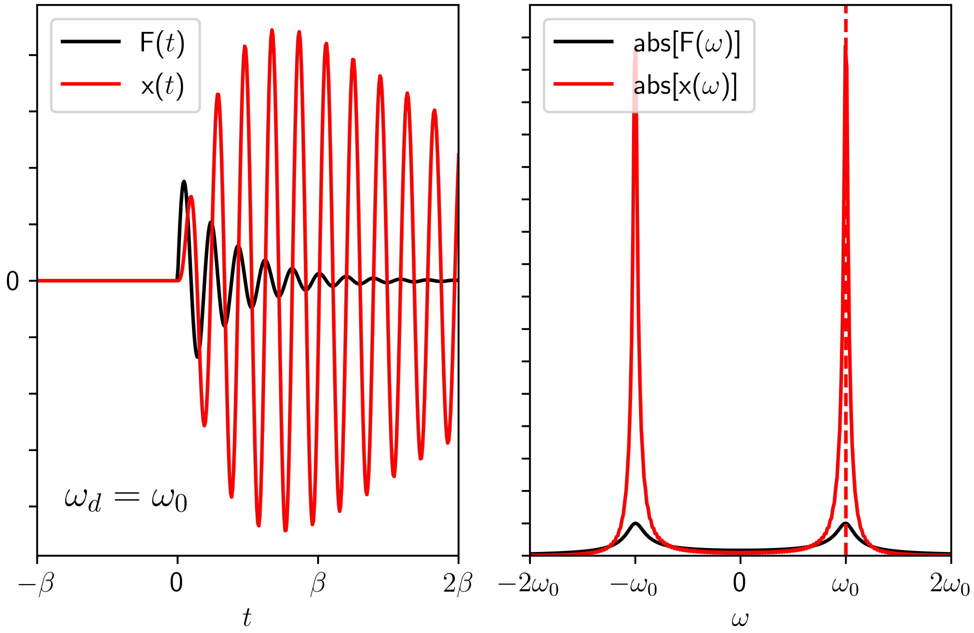

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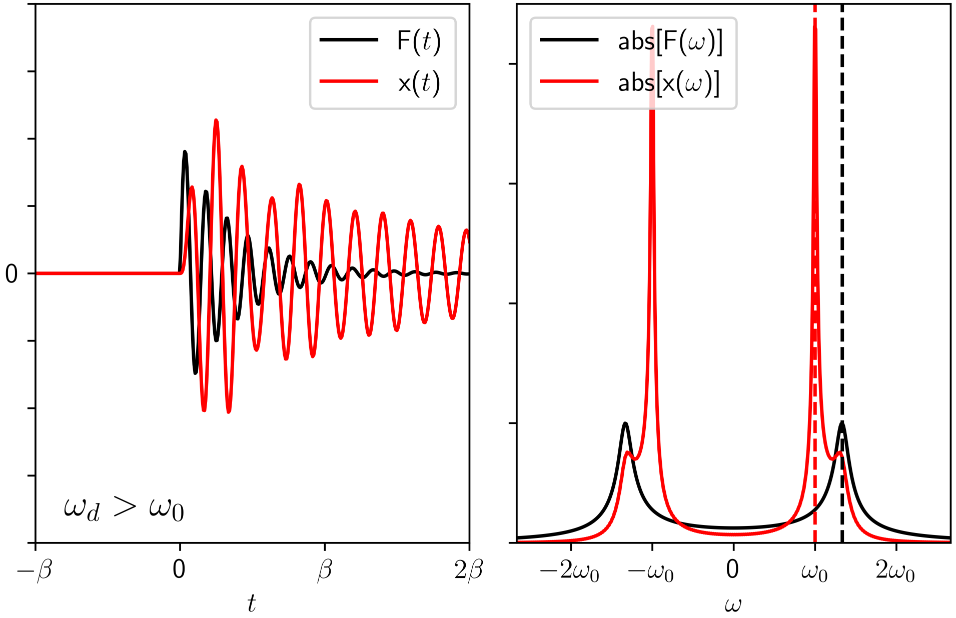

Fourier transform of force and oscillations

$$\mathscr{F}\left[f(t)\right] = \tilde{f}(\omega) = \int_{-\infty}^{+\infty} f(t)e^{-i\omega t}dt$$

Green's functions

- We want to **connect solutions**

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- One method: we can use a **Green's function**!

$$x(t) = \int_{-\infty}^{+\infty} G(t-t')F(t')dt'.$$

Green’s functions

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- One method: we can use a Green’s function!

$$x(t) = \int_{-\infty}^{+\infty} G(t-t')F(t')dt'.$$

- Fourier transforms turn **convolutions** into **products**:

$$\tilde{x}(\omega) = \tilde{G}(\omega)\tilde{F}(\omega).$$

$$\tilde{G}(\omega) = \frac{-1/m}{(\omega^2 - \omega_0^2) - 2i\beta\omega}$$

Green's function of the DDO

$$\tilde{x}(\omega) = \tilde{G}(\omega)\tilde{F}(\omega).$$

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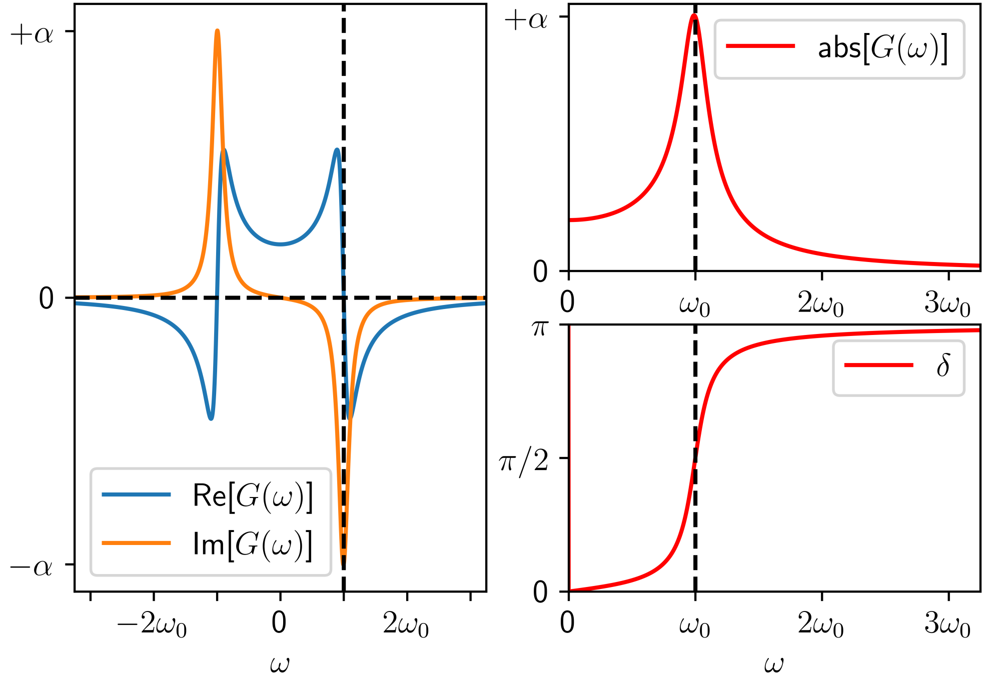
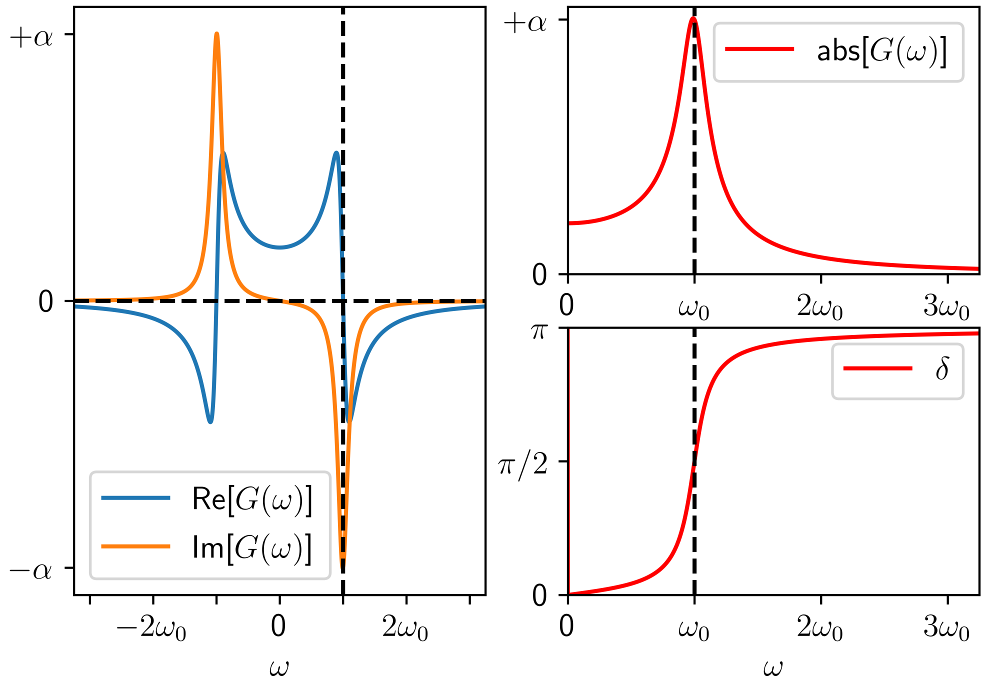


Observations re: the DDO Green's function

- **Amplitude greatest near ω_0**
- $G(\omega)$ produces position oscillations ***in phase*** with the force at **low frequency**
- $G(\omega)$ produces position oscillations ***out of phase*** with force at **high frequency**



$$\tilde{x}(\omega) = \tilde{G}(\omega)\tilde{F}(\omega).$$

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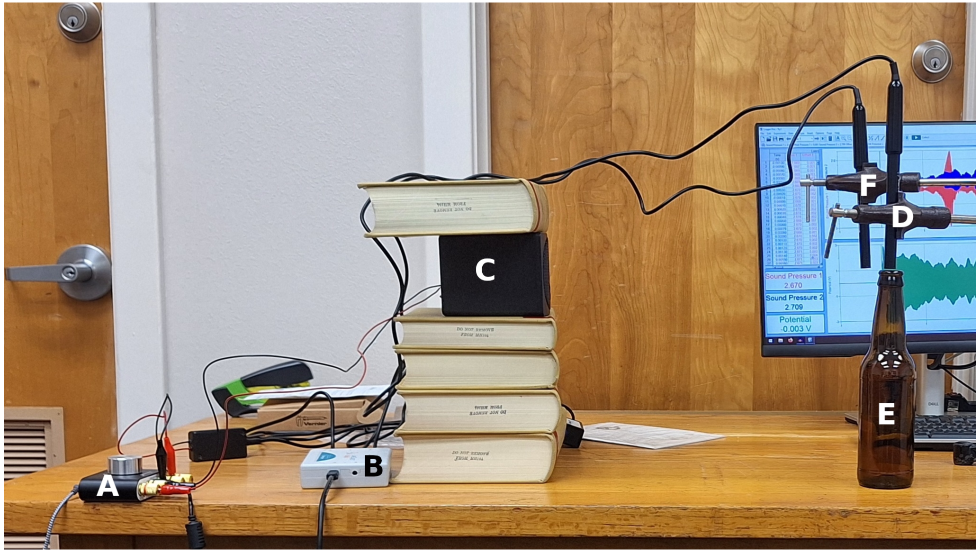
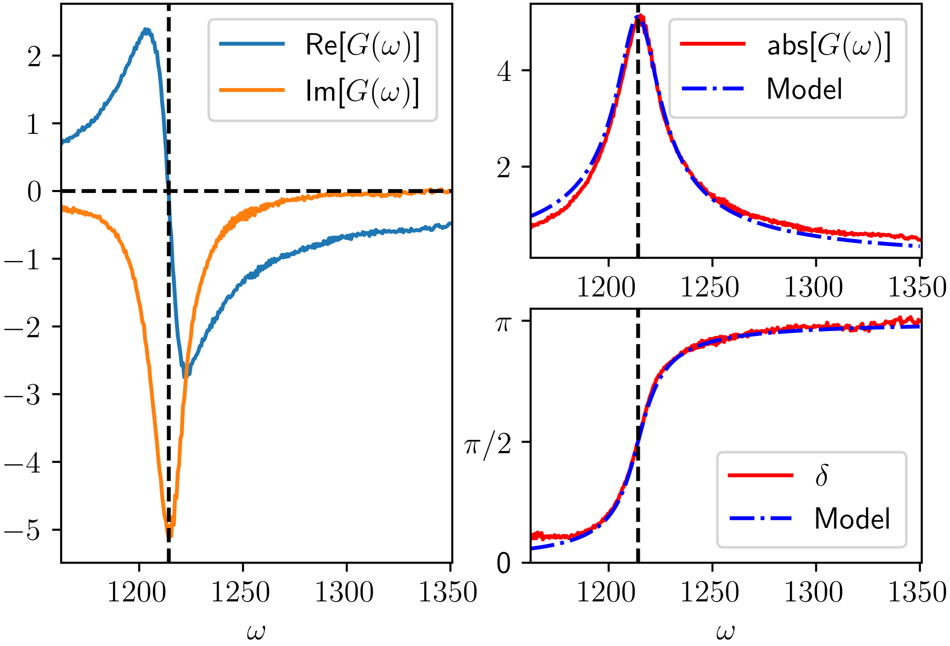
Measuring the acoustical Green's function of a beer bottle

- **Pressure oscillations** above a beer bottle **act like DDOs**
- Could we **measure** their associated **Green's function**?





Background reading

An undergraduate experiment demonstrating the physics of metamaterials with acoustic waves and soda cans

James T. Wilkinson; Christopher B. Whitehouse; Rupert F. Oulton; Sylvain D. Gennaro

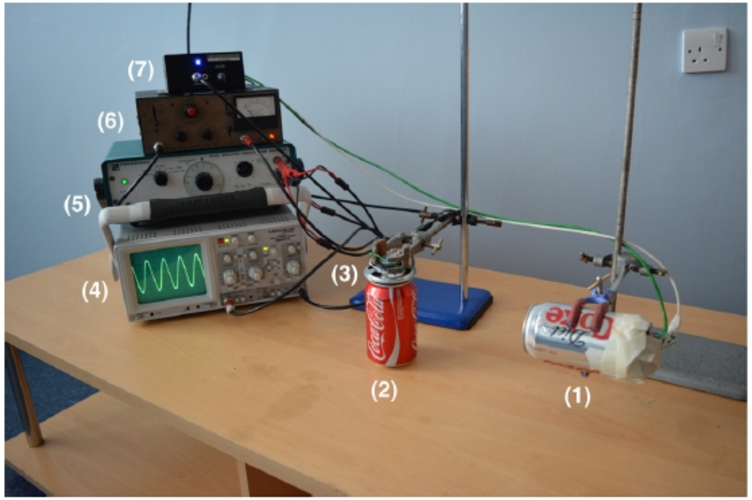
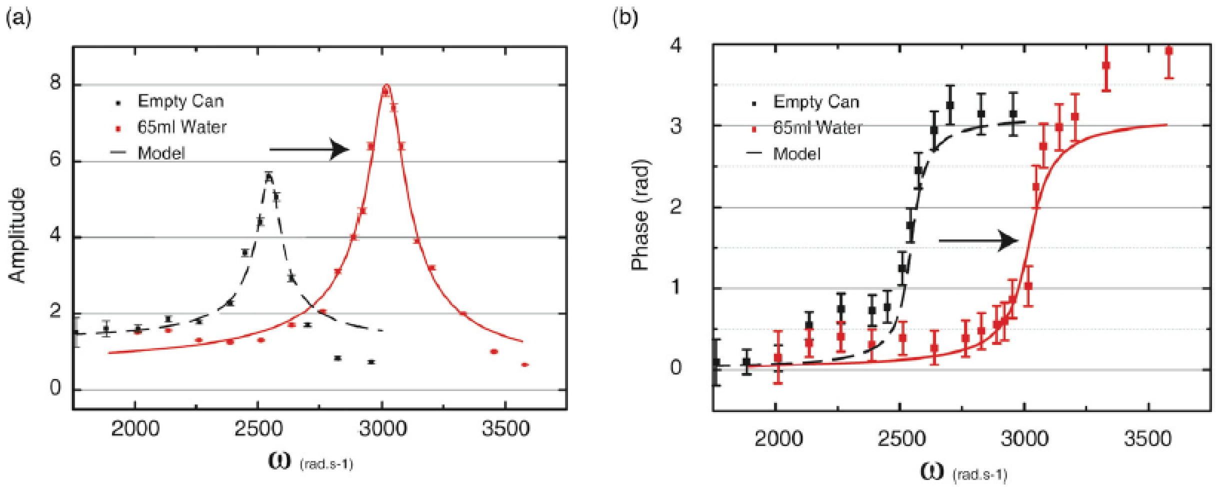


Fig. 2. Experimental apparatus: (1) source (can plus loudspeaker); (2) can under investigation; (3) receiver (loudspeaker used as microphone); (4) oscilloscope; (5) frequency generator; (6) microphone amplifier; (7) source amplifier.



Helmholtz resonators

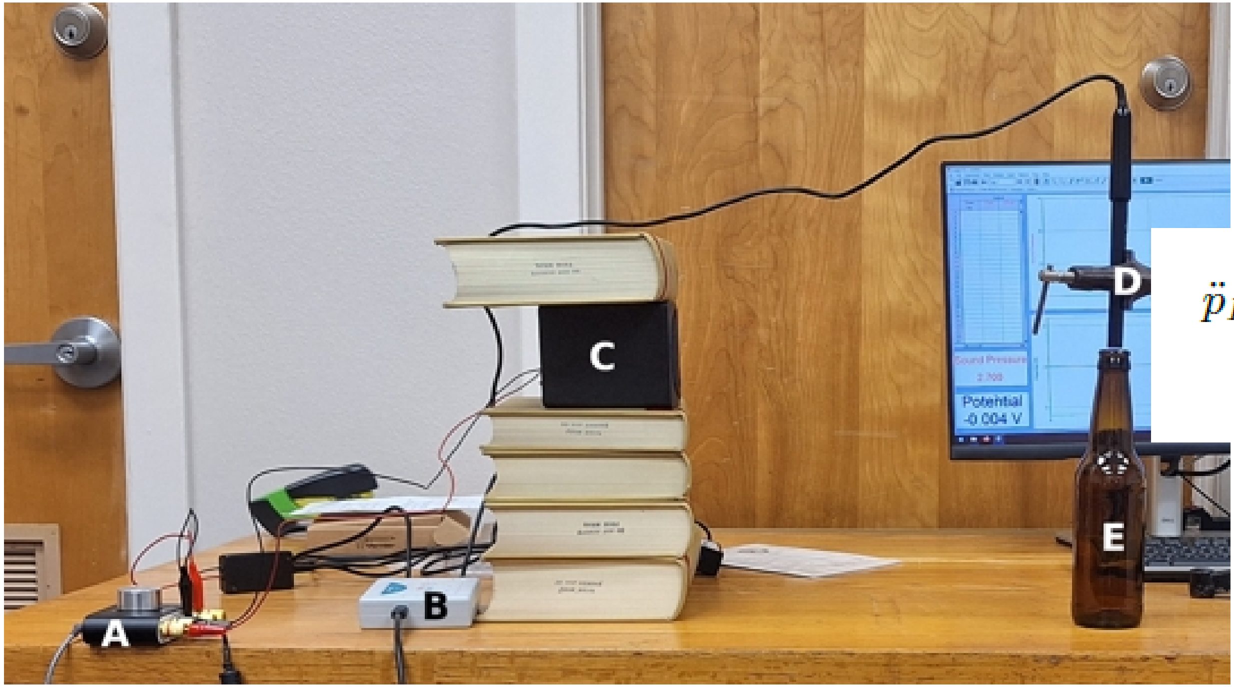
- Imagine an **oscillating plug of air** in the resonator's opening
- The **force** on the air plug depends on…
  - **area** of the opening
  - internal **volume** of the resonator
- This yields a prediction for ω_0:

$$\omega_0^2 \approx \frac{\gamma P_A A^2}{m V_0}.$$

https://en.wikipedia.org/wiki/Helmholtz_resonance

Helmholtz resonators as DDOs

- *Even if* the beer bottle does not act as an ideal Helmholtz resonator, the damped driven oscillator should basically apply



- Introduces resonance, damping, and coupling parameters

$$\ddot{p}_B(t) = -\underbrace{\omega_0^2 p_B(t)}_{\substack{\text{restoring}\\\text{force}}} + \underbrace{2\alpha\beta\omega_0 p_S(t)}_{\substack{\text{driving}\\\text{force}}} - \underbrace{2\beta\dot{p}_B(t)}_{\substack{\text{damping}\\\text{force}}}.$$

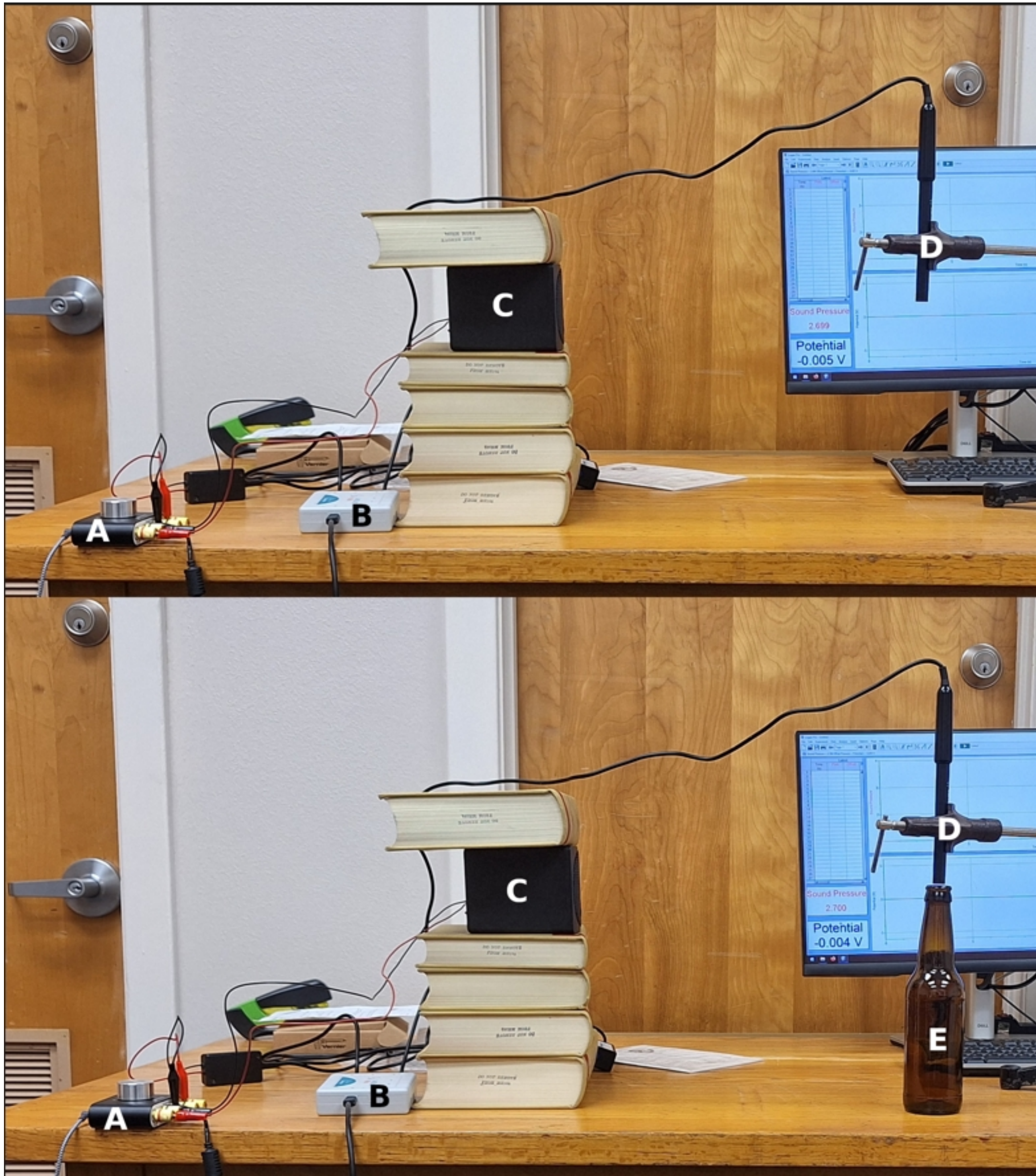
Setup

- First, record sound *without* the bottle:

$$\underbrace{p_S(t)}_{\text{speaker}}$$

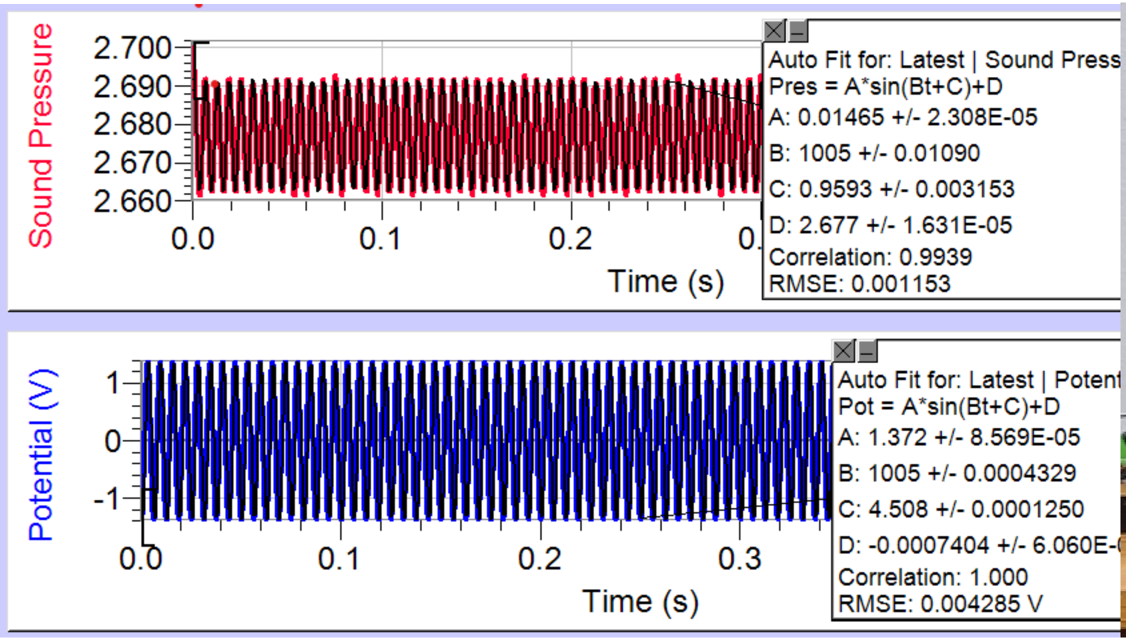
- Next, record sound *with* the bottle to find its contribution:

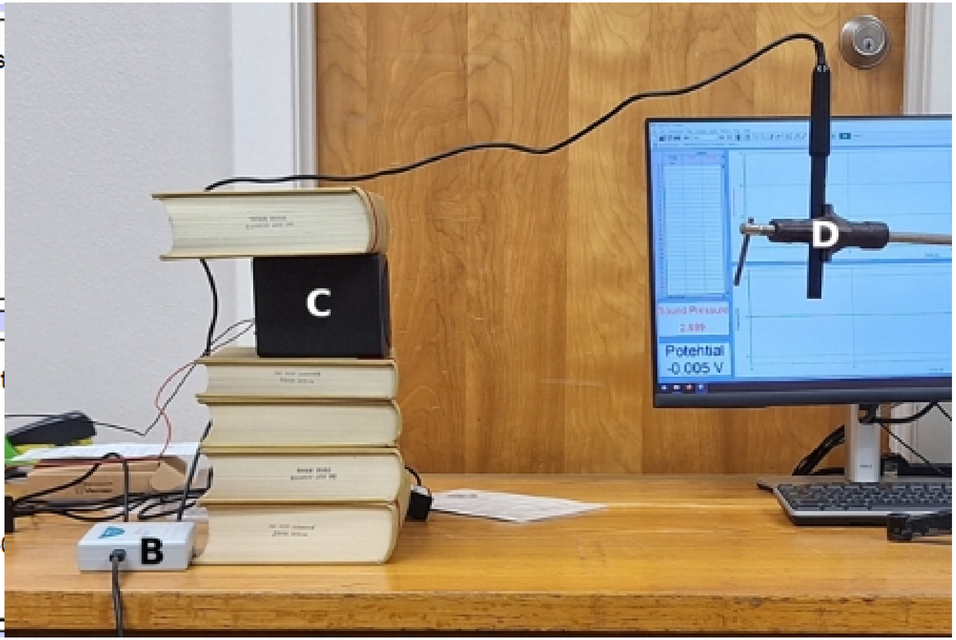
$$\underbrace{p_M(t)}_{\text{microphone}} = \underbrace{p_S(t)}_{\text{speaker}} + \underbrace{p_B(t)}_{\text{bottle}}.$$



Experimental signal

- Easy to measure signal amplitude with and without the bottle
- By monitoring the speaker, we can also measure the phase shift





Normalizing the signal + extracting fit parameters

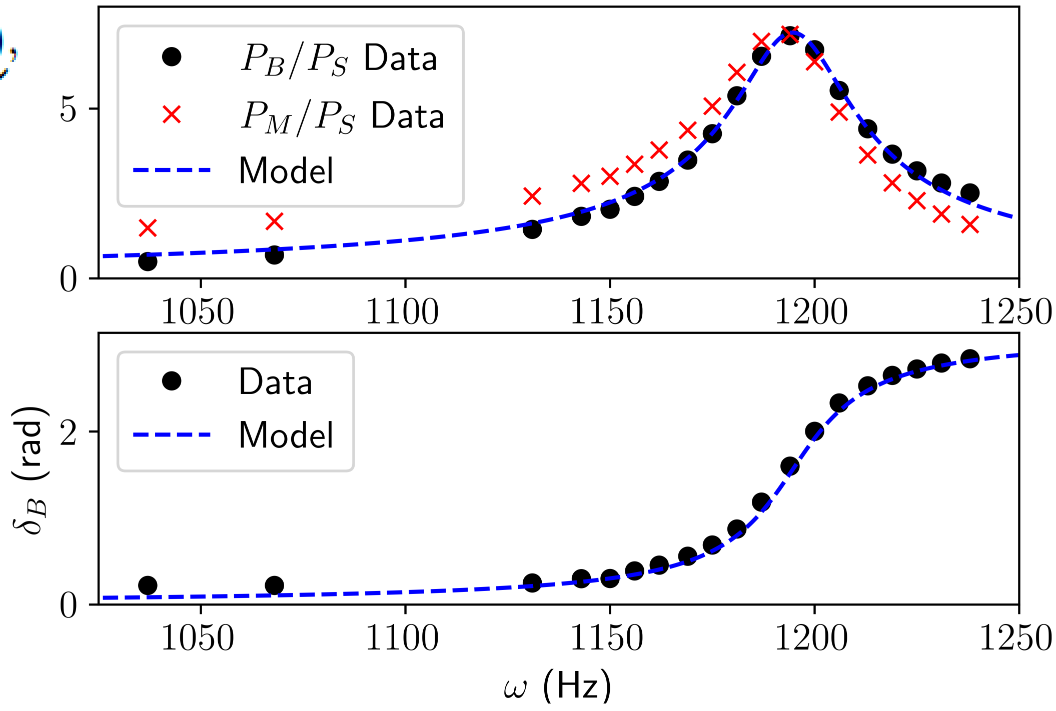
- For pure tones,

$$\underbrace{P_S \cos(\omega t)}_{\text{speaker}} + \underbrace{P_B \cos(\omega t - \delta_B)}_{\text{bottle}} = \underbrace{P_M \cos(\omega t - \delta_M)}_{\text{microphone}},$$

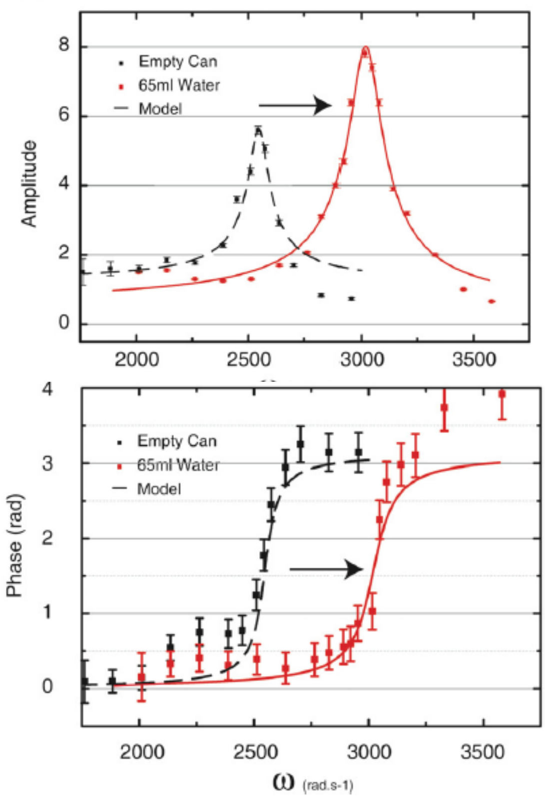
$$P_S e^{i\omega t} + P_B e^{i(\omega t - \delta_B)} = P_M e^{i(\omega t - \delta_M)}$$

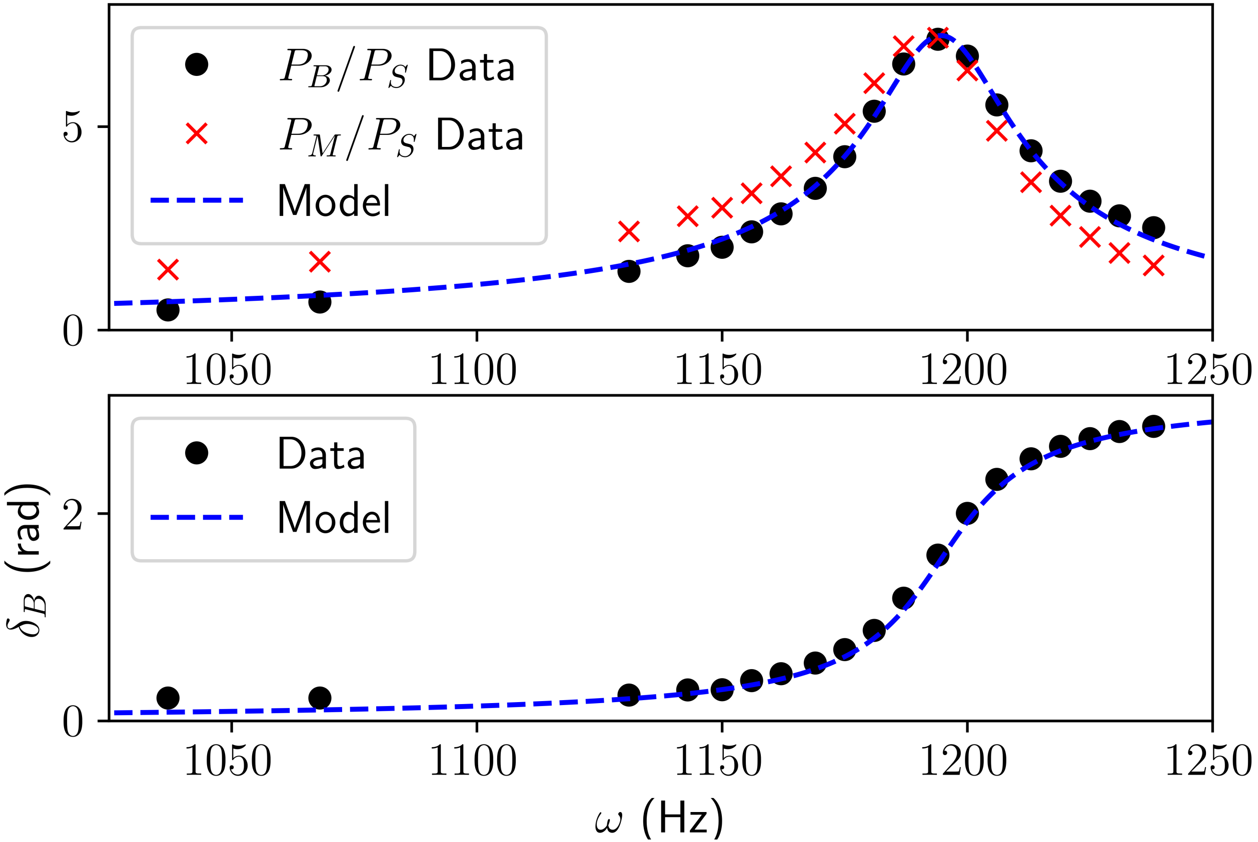
- Normalized amplitudes and phase shifts are fit well by DDO model

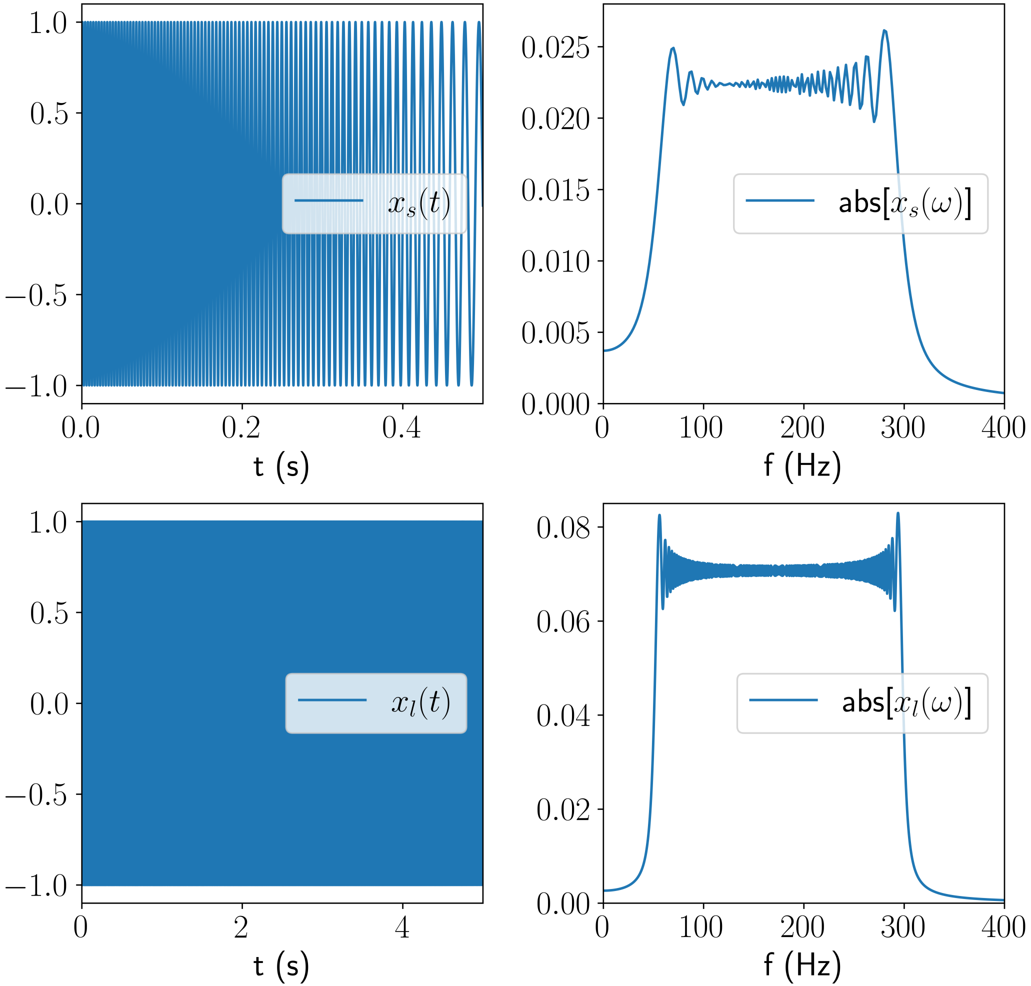
$$P_B = \sqrt{P_S^2 + P_M^2 - 2 P_S P_M \cos(\delta_M)}$$



Big improvement over previous fit!







Chirp signals

- **Chirp signals** are easy to generate using Audacity

$$x(t) = \sin\left[2\pi\left(\frac{c}{2}t^2 + f_0 t\right)\right]$$

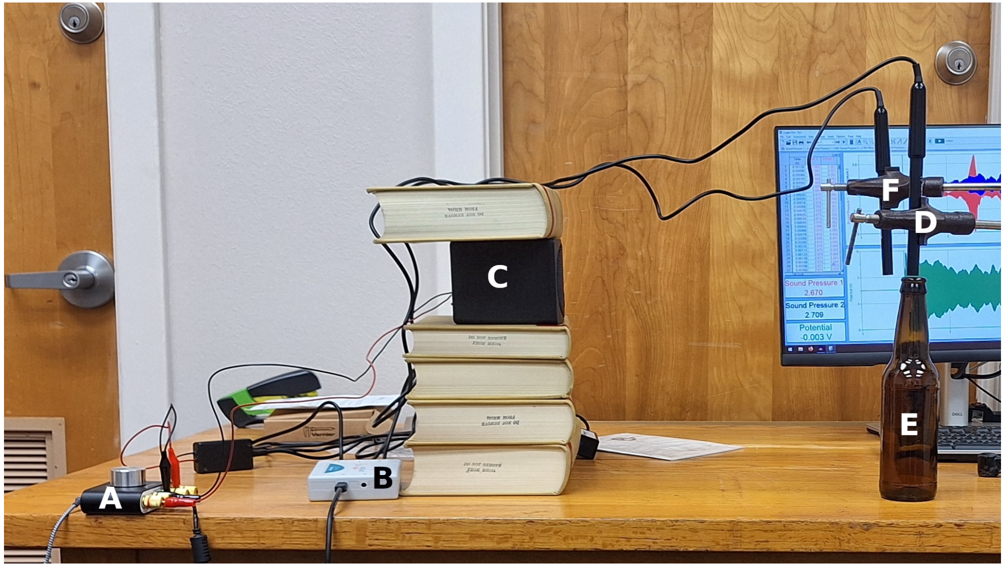
$$c = \frac{f_1 - f_0}{T}.$$

- Chirp spectrum ranges from f_0 to f_1 with a flattish amplitude

Alternative experimental setup

- **Use a chirp signal** to extract DDO parameters efficiently

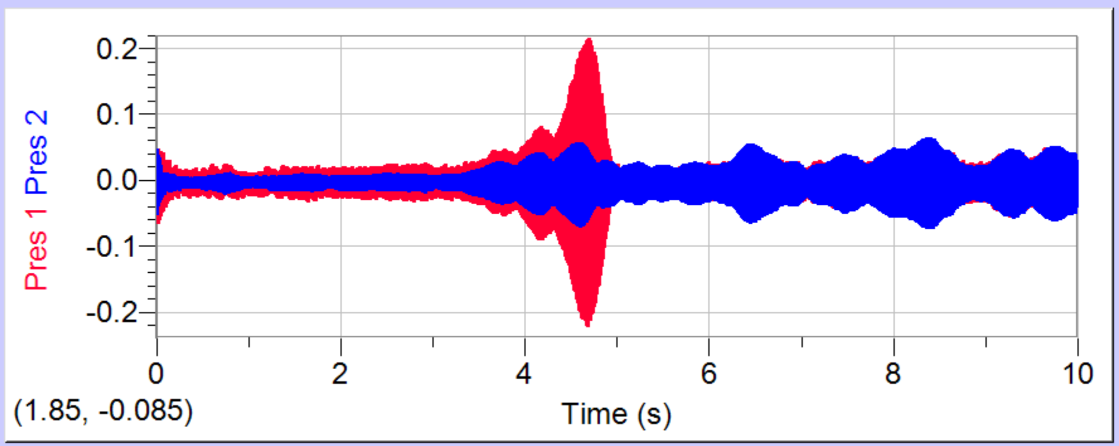
- An approximation: **both mics** are driven by the **same background signal**, up to an unknown time shift

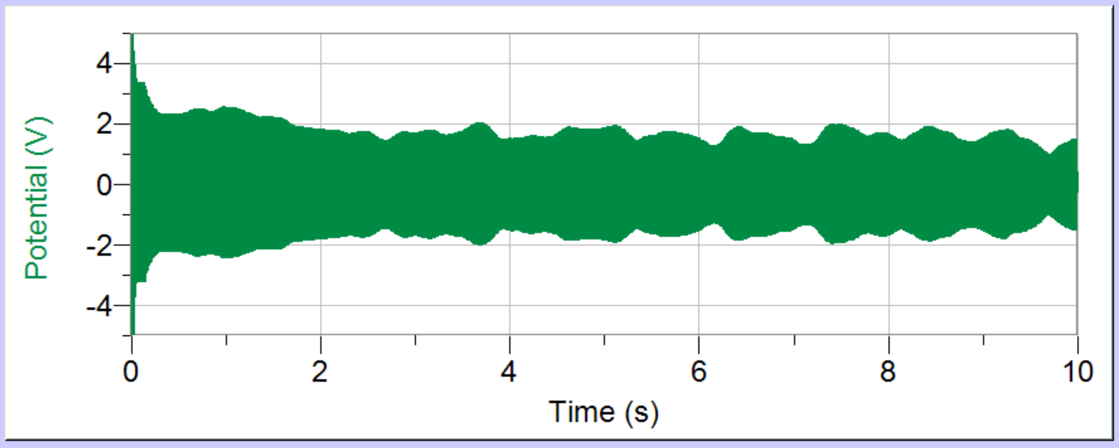


Using the chirp: Raw signal/response

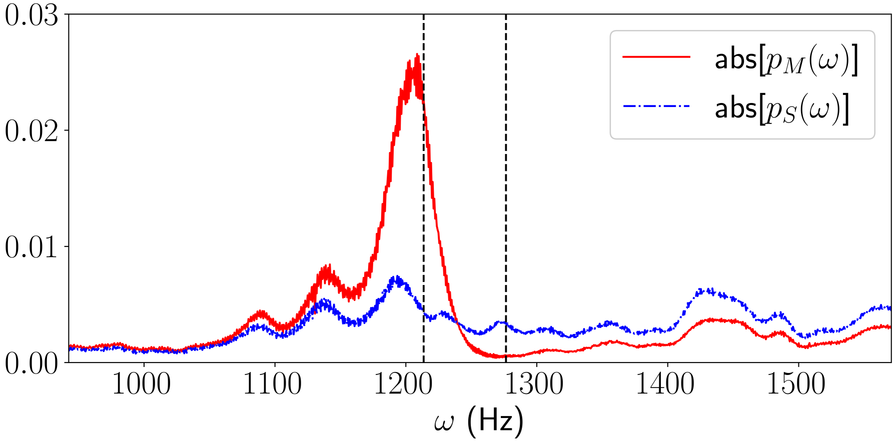
- **Blue**: signal from the mic without a bottle
- **Red**: signal from the mic with a bottle

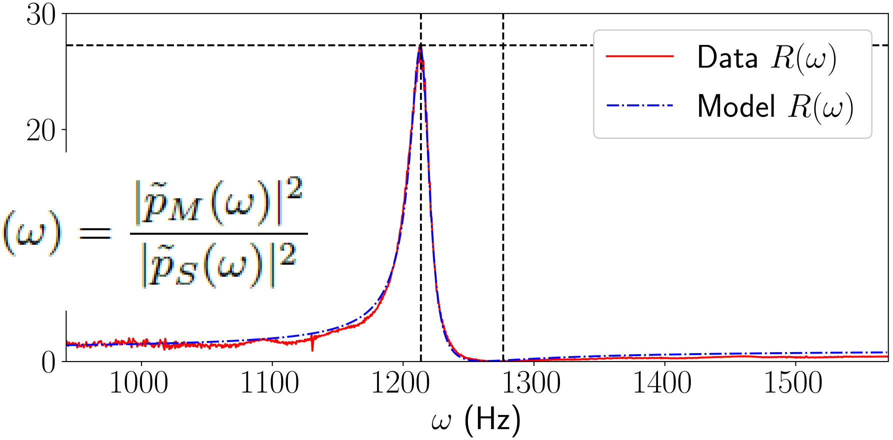
- **Green**: voltage driving the passive speaker





Using the chirp: Normalized signal, incoherent version





$$R(\omega) = \frac{|\tilde{p}_M(\omega)|^2}{|\tilde{p}_S(\omega)|^2}$$

- The DDO model gives a straightforward prediction for the **squared ratio of the magnitudes of the spectra with and without the bottle present**

$$R(\omega) = \frac{(2\alpha\beta\omega_0 + \omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}$$

- This function can be fit to data to extract model parameters

Using the chirp: Normalized signal, coherent version

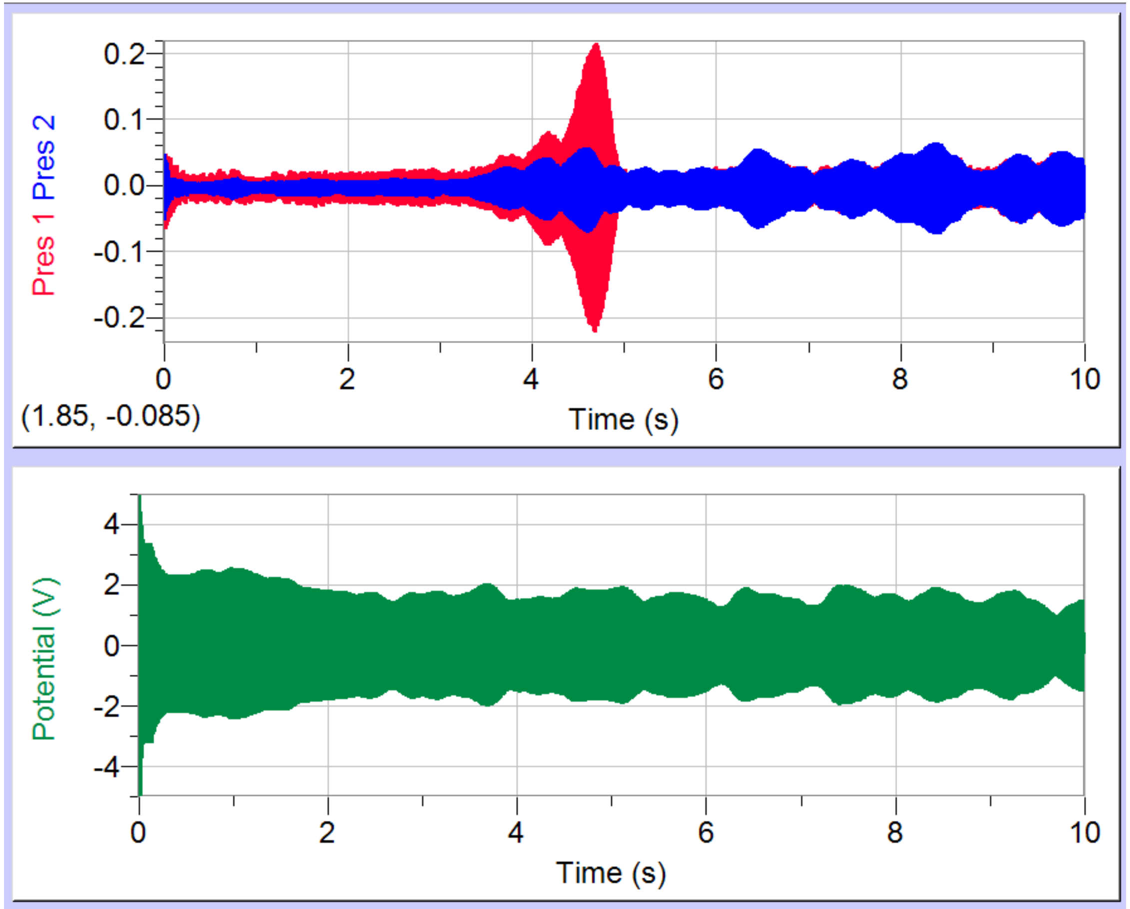
- Suppose the signals give us what we want, up to an undetermined time shift Δ

$$p_1(t) = p_M(t)$$
$$p_2(t) = p_S(t + \Delta)$$

- The spectrum of their *convolution* can lead us to the Green's function!

$$F(\omega) = \frac{\mathscr{F}\left[p_2(-t) * p_1(t)\right]}{|\tilde{p}_2(\omega)|^2}.$$

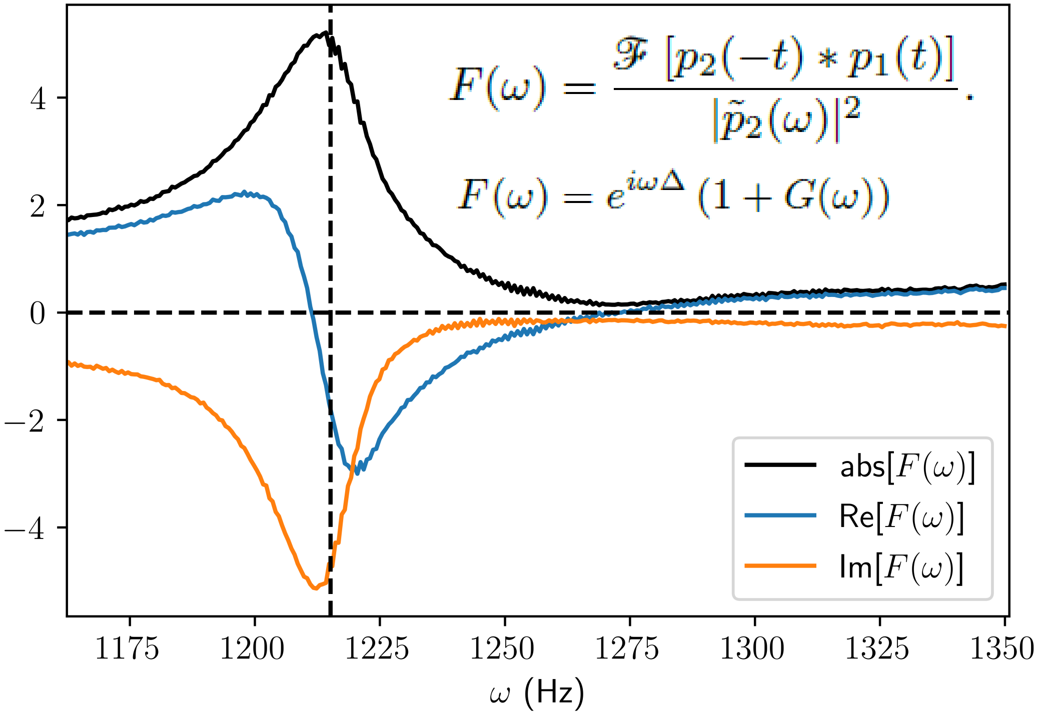
$$F(\omega) = e^{i\omega\Delta}\left(1 + G(\omega)\right)$$



Estimating time shift

- Estimate ω_0 as the max of $|F(\omega)|$
- Extract $|F(\omega_0)|$
- Use the DDO form of $G(\omega_0)$ to estimate Δ:

$$\Delta \approx \frac{1}{i\omega_0} \log\left(\frac{F(\omega_0)}{1 - i\sqrt{|F(\omega_0)|^2 - 1}}\right)$$

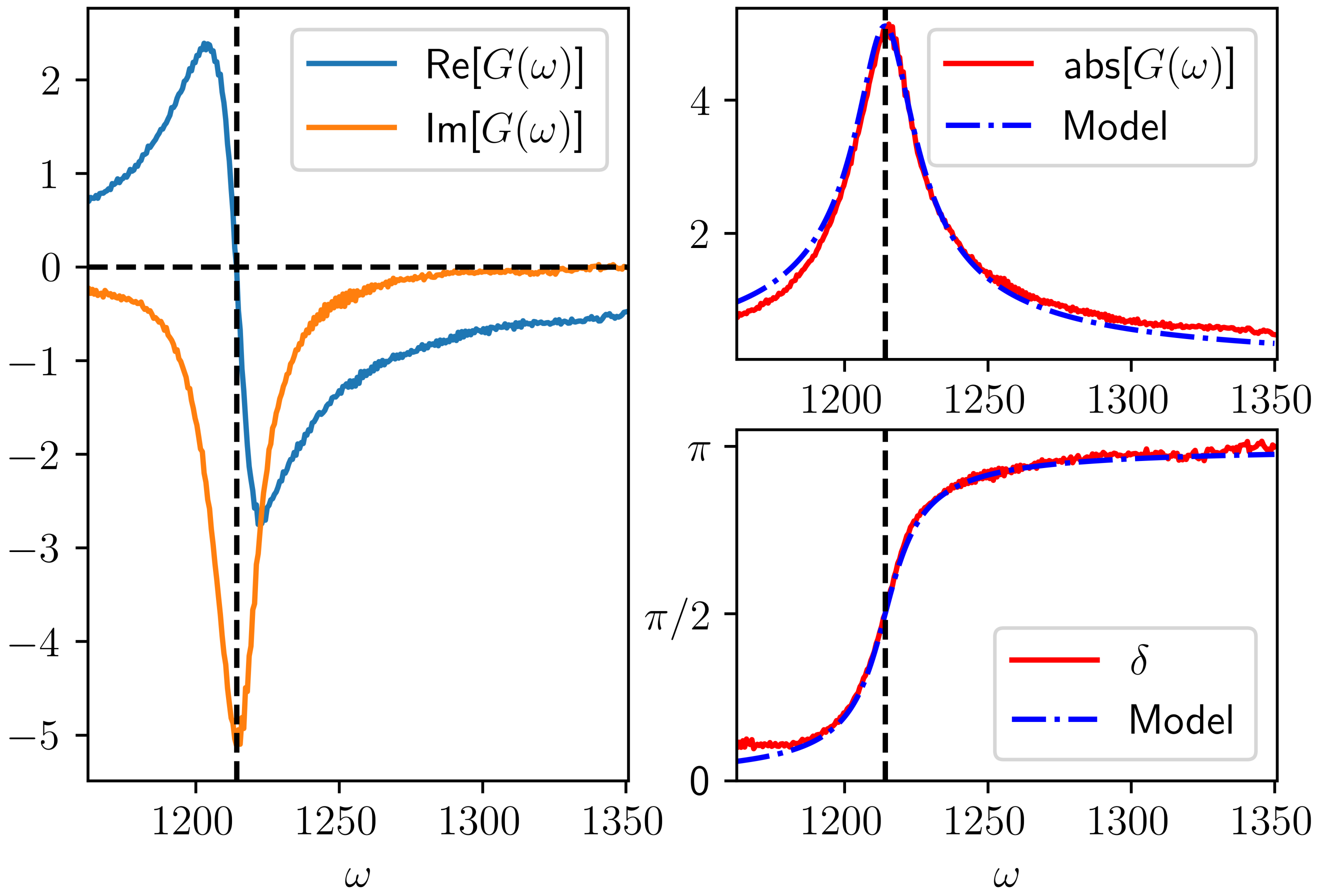


Extracting Green's function from coherent measurement

- Using the Δ estimate, we calculate $G(\omega)$

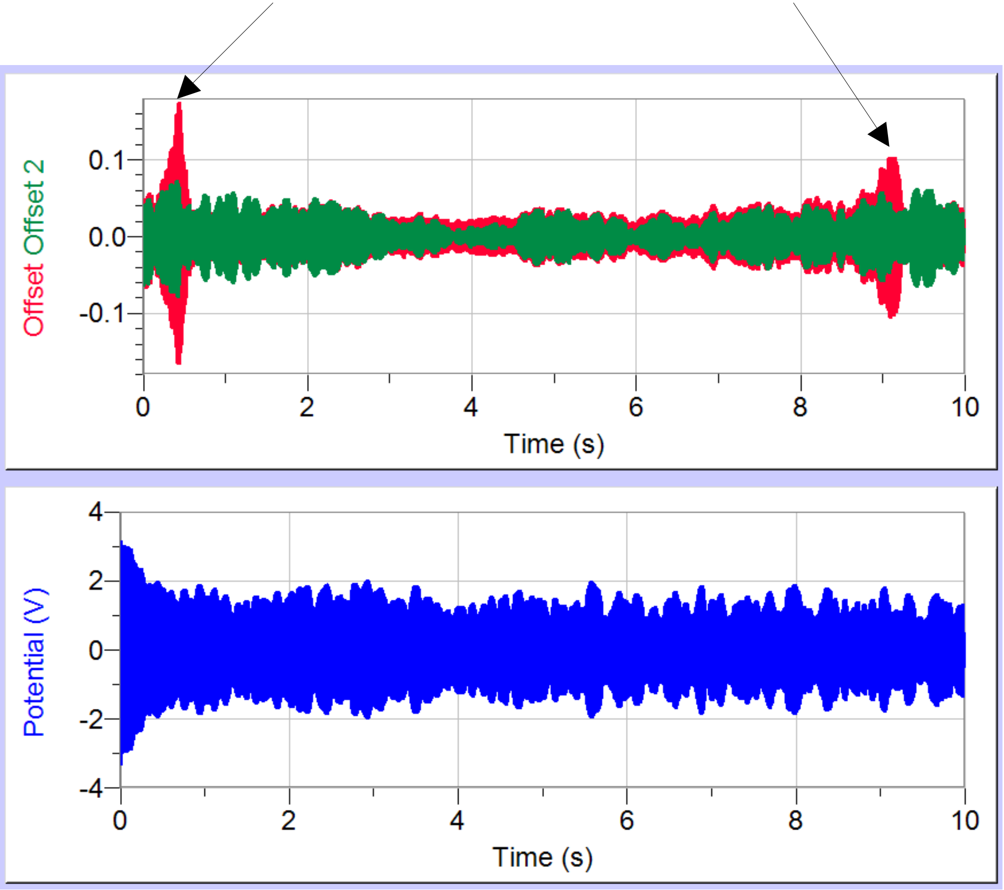
$$G(\omega) \approx e^{-i\omega\Delta} F(\omega) - 1$$

- Gives us both phase and magnitude!



Possible extension

- A single damped oscillator does not capture **higher harmonics**—but seems plausible as a next step for modeling

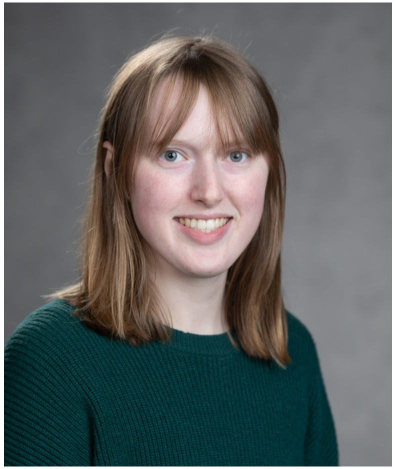


Summary

- The damped driven oscillator can be used to model pressure variations above a beer bottle
- Contributions of the bottle and the background signal can be separated by careful modeling
- The frequency-domain Green's function of the bottle matches the DDO Green's function quite well

Thanks!

- Emma Foster
  - Junior at Centenary who carried out this experiment



- Centenary Faculty-Student Summer Research Grant
  - Paid for this experiment