Entanglement in a model of inelastic scattering

Dr. David Kordahl

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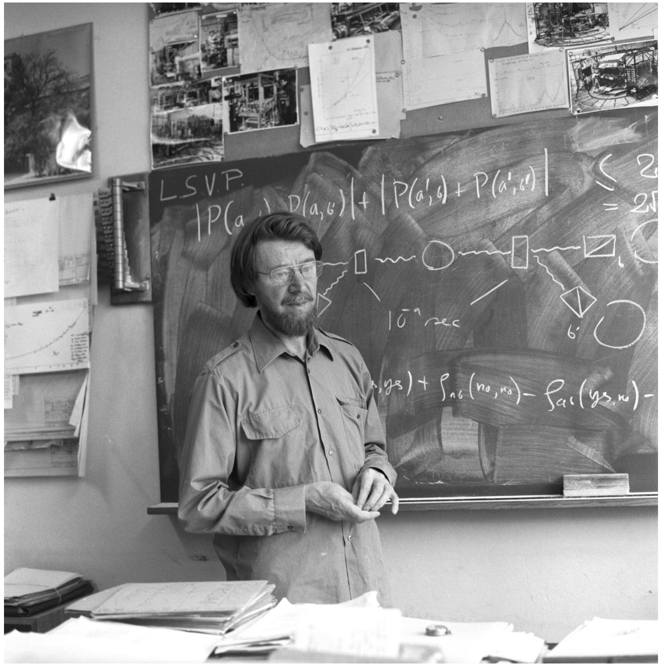
Centenary College of Louisiana

Outline

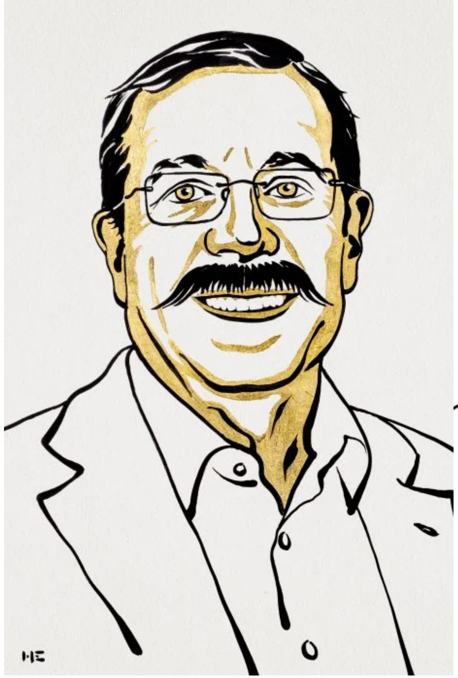
- Reminder of typical entanglement presentations
  - Entanglement and spins
  - Entanglement – “not just for spins”
- Entanglement in a 1D model of inelastic scattering
  - Classical oscillator, classical beam
  - Quantum oscillator, classical beam
  - Quantum oscillator, quantum beam
- Conclusion

2022 Nobel Prize

- Citation: “for experiments with entangled photons, establishing the violation of Bell inequalities and pioneering quantum information science.”

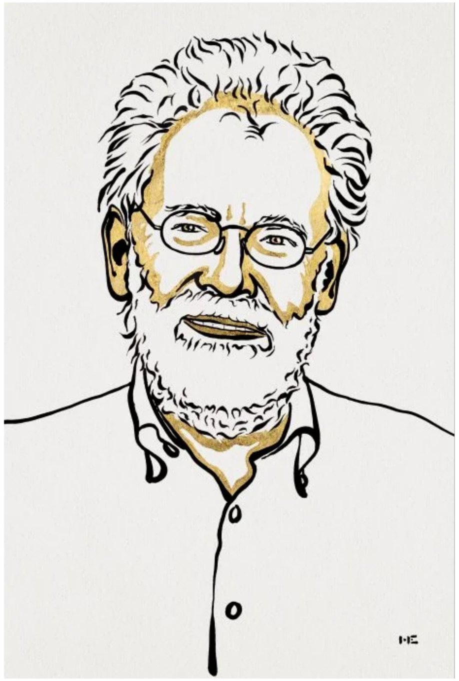


John S. Bell
(1928-1990)



Alain Aspect

John F. Clauser



Anton Zeilinger

Bell States

- Most introductions to entanglement involve the spins of spatially separated quantum particles, initially prepared in one of the “Bell States”

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |0\rangle_B + |1\rangle_A \otimes |1\rangle_B)$$

$$|\Phi^-\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |0\rangle_B - |1\rangle_A \otimes |1\rangle_B)$$

$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |1\rangle_B + |1\rangle_A \otimes |0\rangle_B)$$

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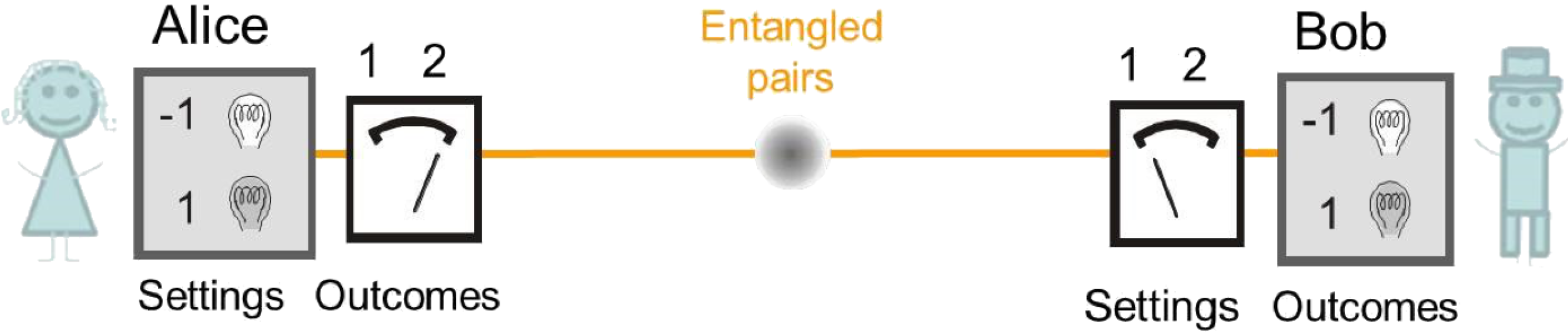
Illustration from the research webpage of Caslav Brukner

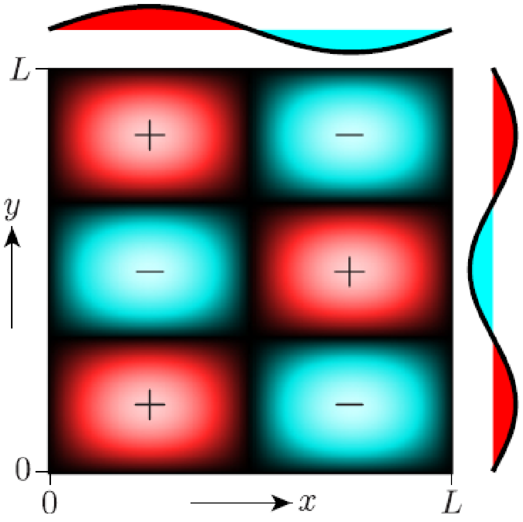


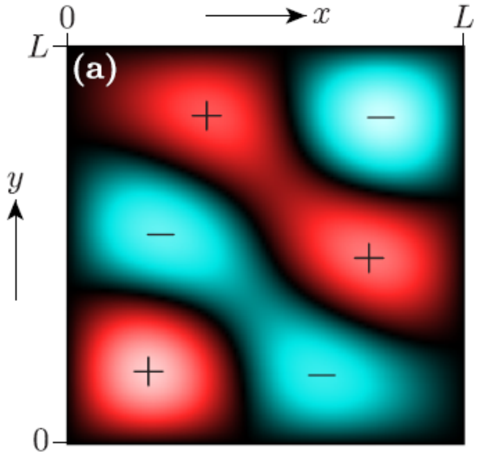
“Entanglement Isn’t Just for Spins”

Daniel V. Schroeder’s article (Am. J. Phys. 85 (11)) explicitly illustrates entanglement for two 1D particles trapped in a box

$$\psi(x,y) \propto \sin\left(\frac{n_x \pi x}{L}\right)\sin\left(\frac{n_y \pi y}{L}\right)$$

$$\psi \propto \sin\left(\frac{2\pi x}{L}\right)\sin\left(\frac{3\pi y}{L}\right) + \frac{1}{2}\sin\left(\frac{3\pi x}{L}\right)\sin\left(\frac{2\pi y}{L}\right)$$





Entanglement and Inelastic Scattering

- Entanglement in scattering situations arises naturally from the interaction between the “beam” and the “target”

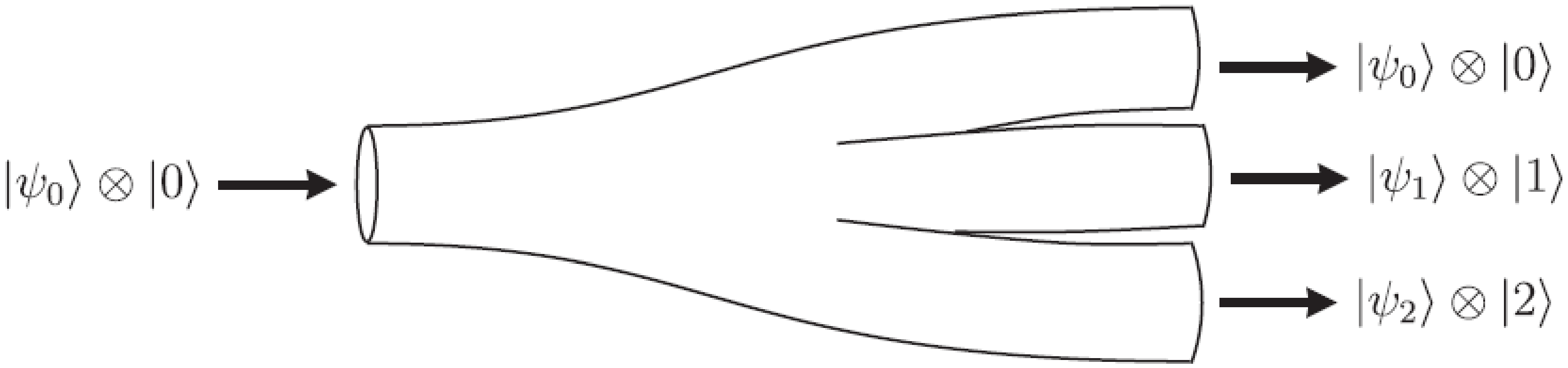
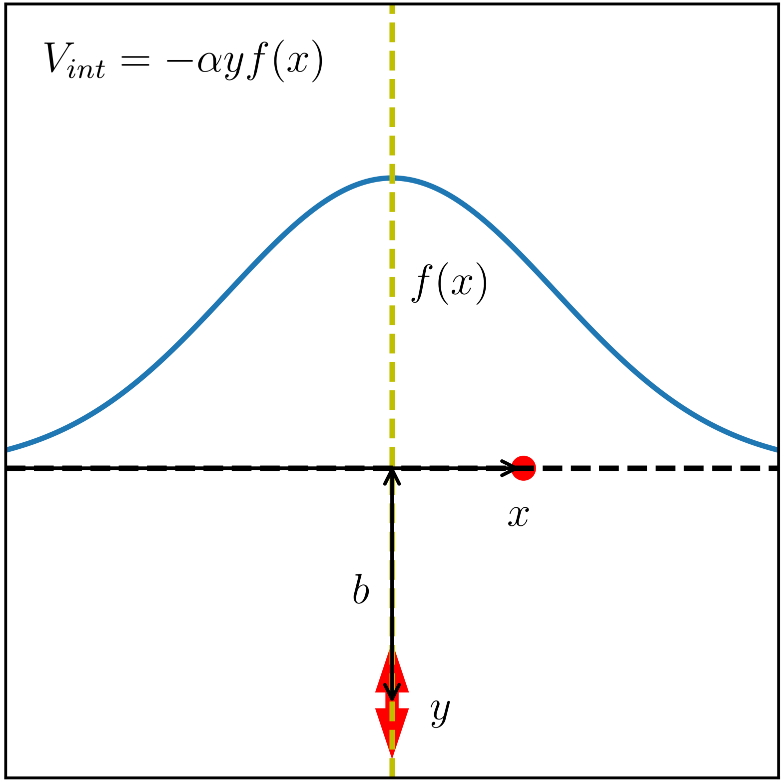


Illustration by Christian Dwyer, in a review article
“Atomic-Resolution Core-Level Spectroscopy
in the Scanning Transmission Electron Microscope”

Simple 1D model of Inelastic Scattering

- Let's capture this with a 1D model:
  - 1D beam (variable: x, mass: m)
  - 1D oscillator (variable: y, mass: μ, angular frequency: ω_0)
- The Hamiltonian is straightforward:

$$H = \underbrace{\frac{p_x^2}{2m}}_{\text{beam}} + \underbrace{\frac{p_y^2}{2\mu} + \frac{1}{2}\omega_0^2 \mu y^2}_{\text{oscillator}} - \underbrace{\alpha\, y\, f(x)}_{\text{interaction}}$$



Consequences of the Full Classical Model

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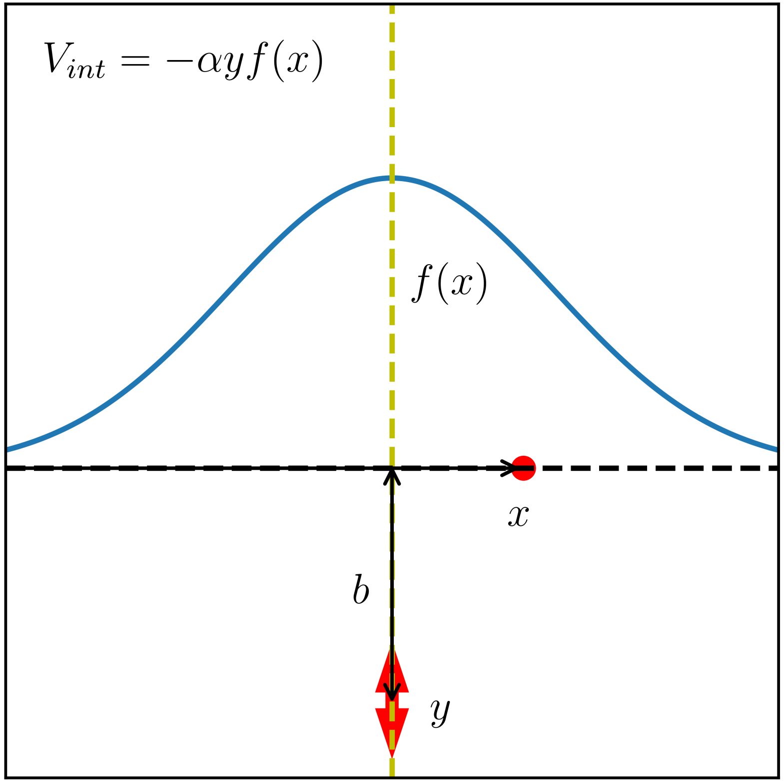
- We can find classical equations of motion for each variable

$$m\ddot{x} = \alpha y \frac{df}{dx}$$

$$\mu\ddot{y} = -\mu\omega_0^2 y + \alpha f(x)$$



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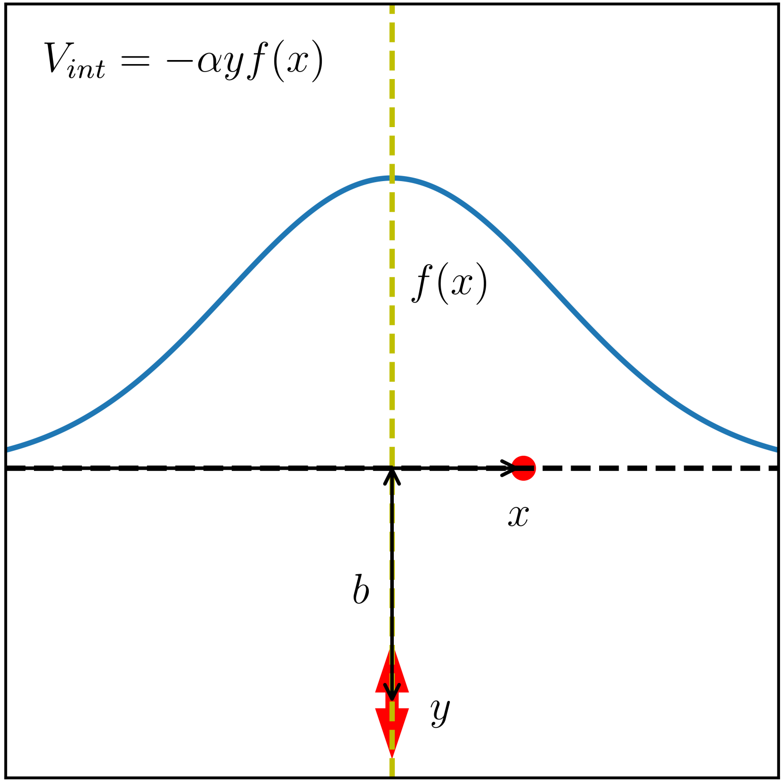
- When the beam travels swiftly ($x \approx v_0 t$), a predictable amount of energy is transferred from beam to oscillator (assuming it starts at rest)

$$W = \frac{\pi\alpha^2}{\mu}\left|\tilde{f}(\omega_0)\right|^2$$

$$\tilde{f}(\omega) = \frac{1}{\sqrt{2\pi}}\int dt\, f(vt)e^{i\omega t}dt$$



Classical Oscillations

- Final amplitude depends directly on model and initial conditions

$$H = \underbrace{\frac{p_x^2}{2m}}_{\text{beam}} + \underbrace{\frac{p_y^2}{2\mu} + \frac{1}{2}\omega_0^2 \mu y^2}_{\text{oscillator}} - \underbrace{\alpha\, y\, f(x)}_{\text{interaction}} \longrightarrow \tilde{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int dt\, f(vt) e^{i\omega t} dt \longrightarrow y_m = \frac{\sqrt{2\pi}\alpha}{\mu\omega_0} \left| \tilde{f}(\omega_0) \right|$$

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$$f(x) = b^{-2} e^{-x^2/b^2}$$

$$\longrightarrow y_m = \frac{\sqrt{\pi}\alpha}{\mu\omega_0} \frac{e^{-b^2\omega_0^2/4v^2}}{vb}$$

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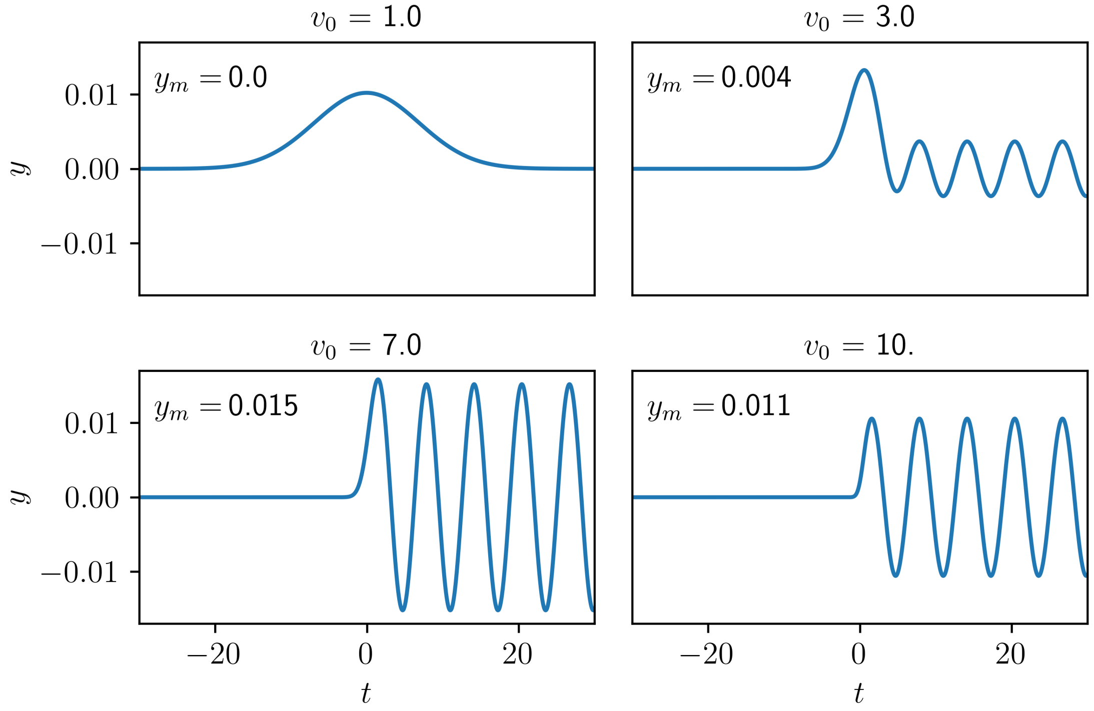
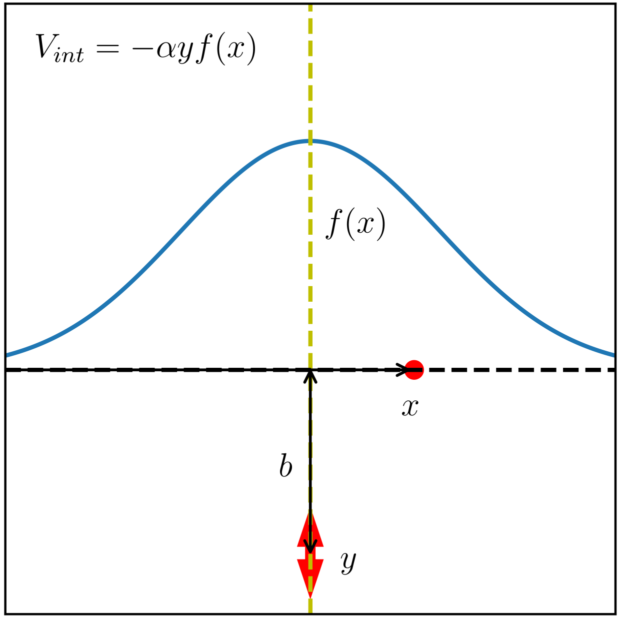
$$\hbar = \omega_0 = m = 1$$
$$b = 10$$
$$\alpha = \mu = 100$$



Quantizing Just the Oscillator

- Problem: Classical approach doesn't reflect experiment – experiment only has *some* beam particles losing energy
- Trial fix: Use a classical beam, and a quantum oscillator

$$H = \underbrace{\frac{p_y^2}{2\mu} + \frac{1}{2}\omega_0^2 \mu y^2}_{\text{unperturbed oscillator}} - \underbrace{\alpha y f(vt)}_{\text{time-dependent perturbation}}$$



Quantizing Just the Oscillator

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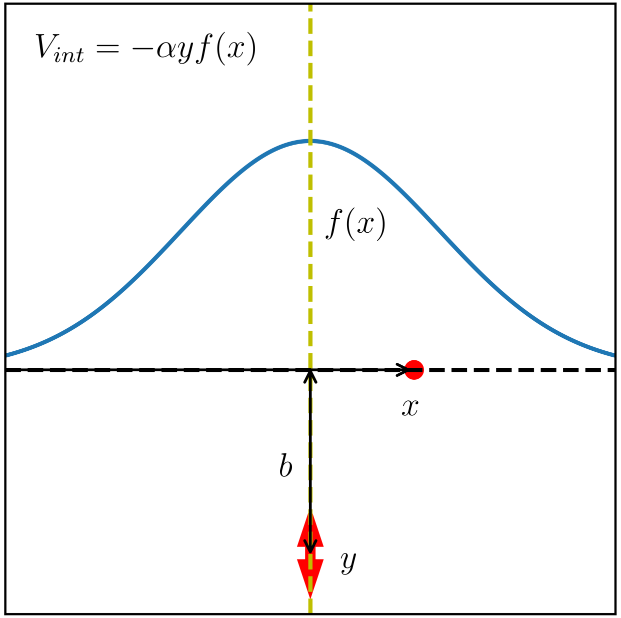
- Trial fix: Use a classical beam, and a quantum oscillator

$$[y, p_y] = i\hbar$$

$$a^\dagger = \frac{1}{\sqrt{2\hbar\omega_0\mu}}(\omega_o\mu y - ip_y)$$

$$a = \frac{1}{\sqrt{2\hbar\omega_0\mu}}(\omega_o\mu y + ip_y)$$

$$H = \underbrace{\hbar\omega_0\left(a^\dagger a + \frac{1}{2}\right)}_{H_0} - \underbrace{\alpha f(vt)\sqrt{\frac{\hbar}{2\omega_0\mu}}(a + a^\dagger)}_{H_{int}(t)}$$



Consequences of Quantizing Just the Oscillator

- Approach 1:
  First-order
  time-dependent
  perturbation theory

$$P_n^{(1)} = \frac{1}{\hbar^2} \left| \int_{-\infty}^{\infty} \langle n | H_{int}(t) e^{in\omega_0 t} | 0 \rangle \, dt \right|^2$$

- To first order, only the
  first energy eigenstate
  may be excited

$$P_1^{(1)} = \frac{\pi \alpha^2}{\hbar \mu \omega_0} |\tilde{f}(\omega_0)|^2$$

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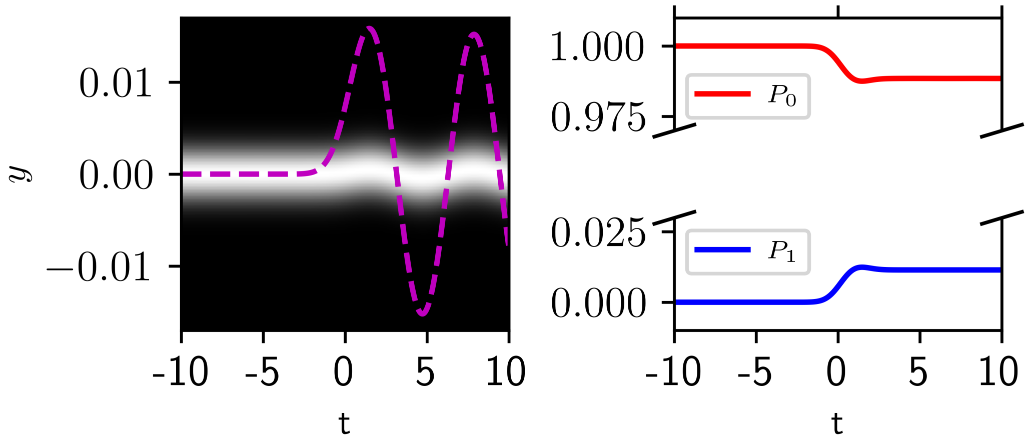
$$P_1^{(1)} = \frac{\pi \alpha^2}{\hbar \mu \omega_0} |\tilde{f}(\omega_0)|^2$$

- Approach 2: Direct integration of the Schrodinger equation

$$i\hbar \frac{\partial}{\partial t} \psi(y,t) = \hat{H}(t) \psi(y,t)$$

classical beam ($v_0 = 7.0$), quantum oscillator



$$\langle y \rangle = \int dy \, \psi^*(y) \, y \, \psi(y)$$

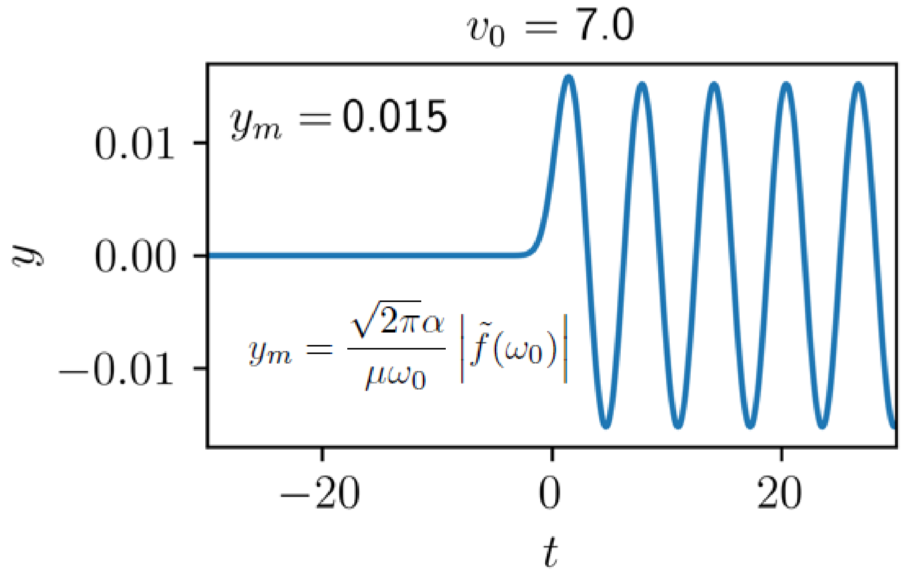
$$\langle \psi_1 | \psi \rangle = \int dy \, \psi_1^*(y) \psi(y)$$

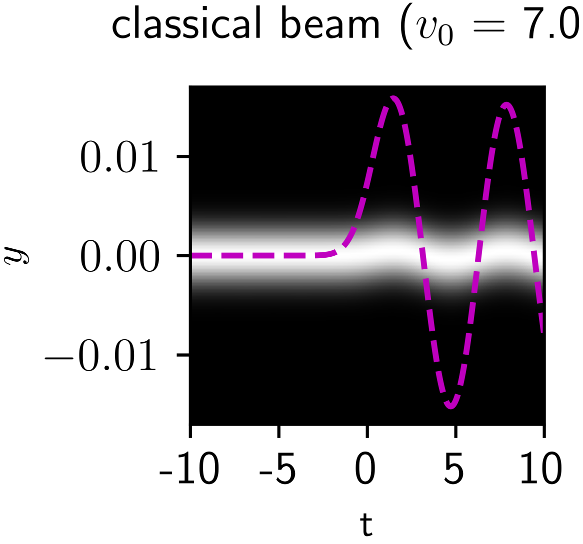
$$P_1 = | \langle \psi_1 | \psi \rangle |^2$$

Classical vs. Partly Quantum Approaches

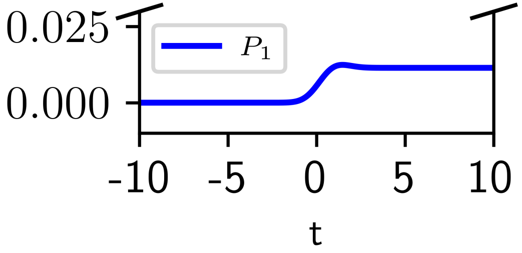
- The classical amplitude of the classical HO oscillator is proportional to the transition amplitude of the quantum HO

$$y_m = \sqrt{\frac{2\hbar}{\mu\omega} P_1^{(1)}} \qquad\qquad P_1^{(1)} = \frac{\mu\omega_0}{2\hbar} y_m^2$$





$$P_1 = \frac{\pi\alpha^2}{\hbar\mu\omega_0}|\tilde{f}(\omega_0)|^2$$



Partly QM vs. Fully QM Approaches

- When will the partly QM approach match the fully QM one?

Partly QM Approach

$$i\hbar\frac{\partial}{\partial t}\psi(y) = \hat{H}(t)\psi(y)$$

$$\hat{H}(t) = -\frac{\hbar^2}{2\mu}\frac{\partial^2}{\partial y^2} + \frac{1}{2}\mu\omega_o^2 y^2 - \alpha y f(vt)$$

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Fully QM Approach

$$i\hbar\frac{\partial}{\partial t}\psi(x,y) = \hat{H}\psi(x,y)$$

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} - \frac{\hbar^2}{2\mu}\frac{\partial^2}{\partial y^2} + \frac{1}{2}\mu\omega_o^2 y^2 - \alpha y f(x)$$

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- Depends on the relation between reduced and full wavefunction

$$\psi(y) = \int dx\, \psi(x,y)$$

Picturing the Fully QM Wavefunction

- Following the partly QM approach, the exact transition probabilities form a Poisson distribution

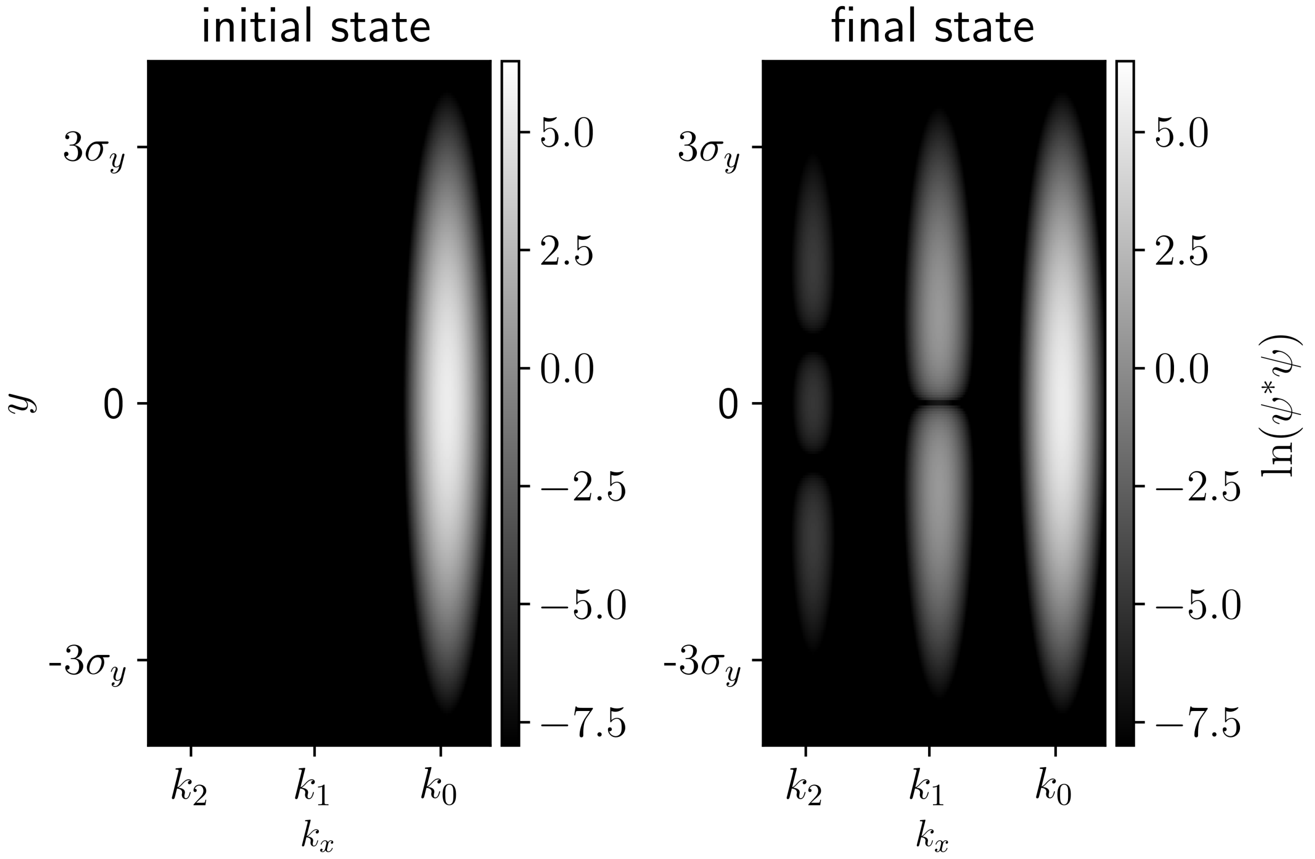
$$P_n = \exp\left(-P_1^{(1)}\right)\frac{\left(P_1^{(1)}\right)^n}{n!}$$

- Following the fully QM approach, our wavefunction would be easier to visualize with the beam in k-space

$$i\hbar\frac{\partial}{\partial t}\tilde{\psi}(k_x, y) = \left(\frac{\hbar^2 k_x^2}{2m} - \frac{\hbar^2}{2\mu}\frac{\partial^2}{\partial y^2} + \frac{1}{2}\mu\omega_0^2 y^2\right)\tilde{\psi}(k_x, y)$$
$$- \frac{\alpha}{\sqrt{2\pi}} y\tilde{f}(k_x) * \tilde{\psi}(k_x, y)$$

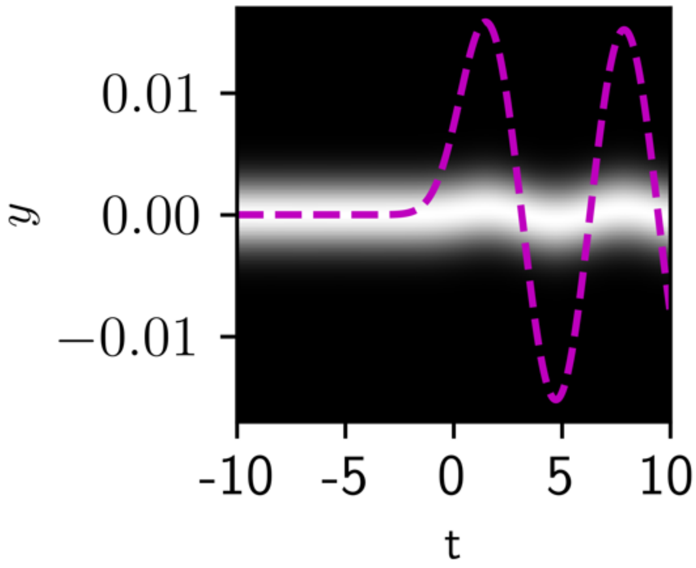
Imagining $\Psi^*\Psi$ in k_x and y

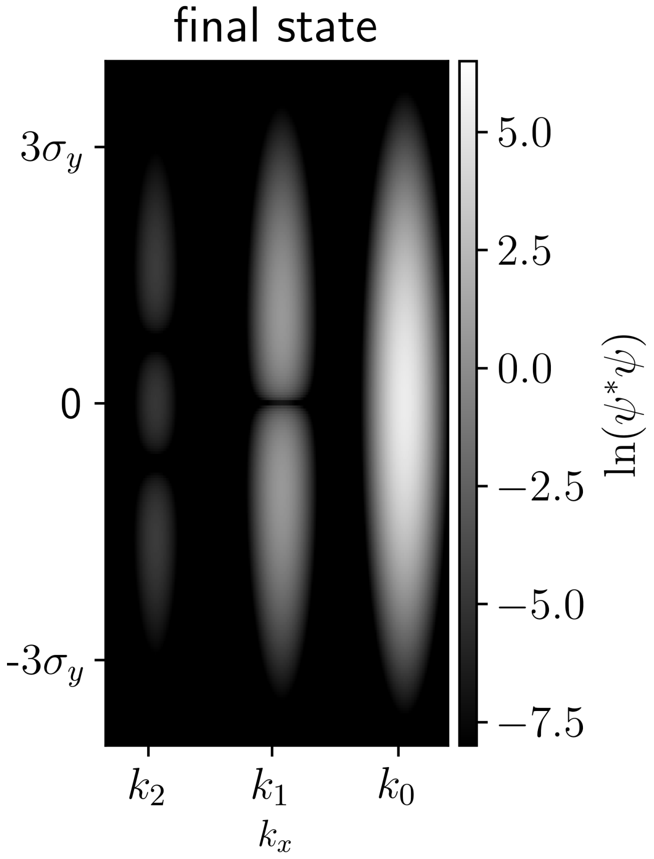
- The probability density of the beam is entangled with that of the oscillator



Imagining $\Psi^*\Psi$ in k_x and y

- The probability density of the beam is entangled with that of the oscillator
- Thinking question: how does this static density relate to the time-dependence of the reduced $\Psi(y)$?

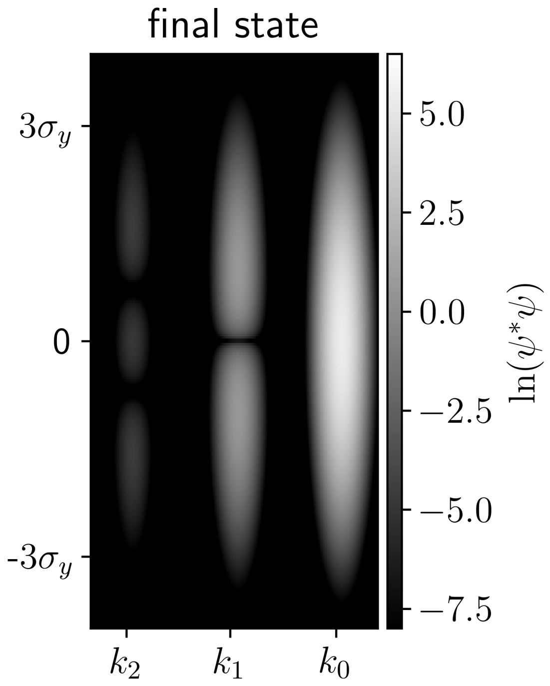




Entanglement between k_x and y

- Measuring k_x changes our predictions for y measurements

$$\psi'(y) \propto \int_{k_n-\Delta_k/2}^{k_n+\Delta_k/2} dk_x\, \psi(k_x, y)$$



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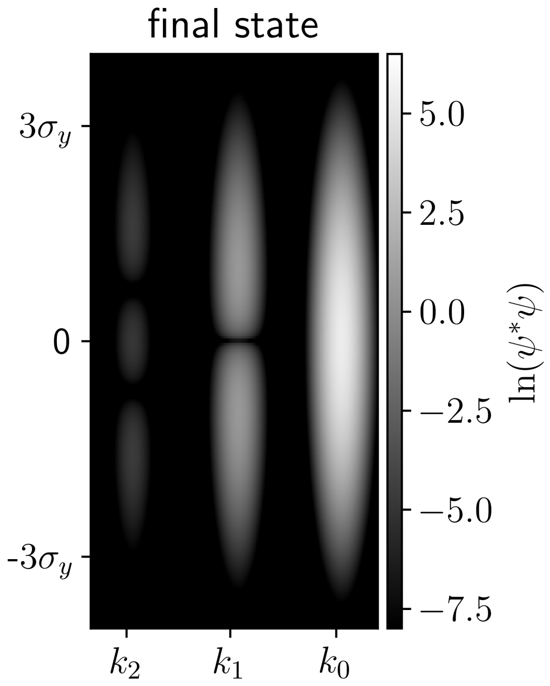
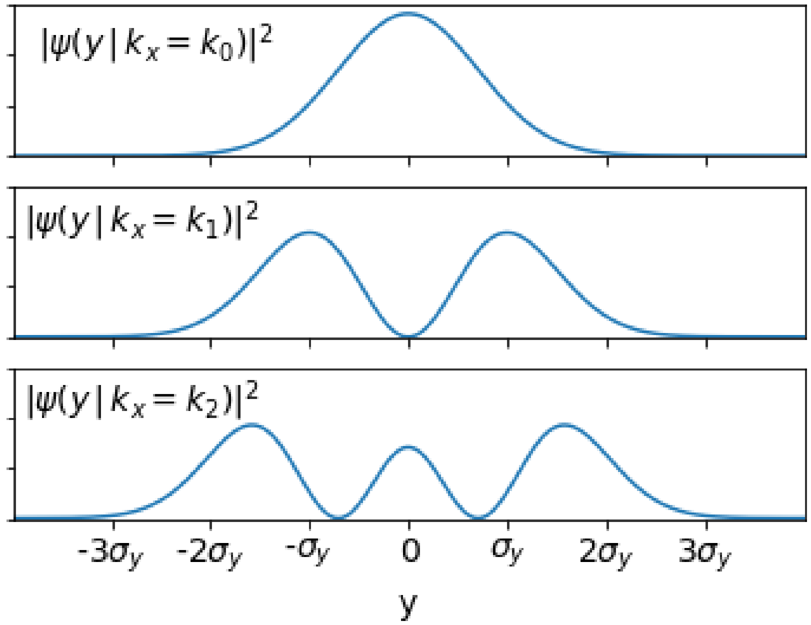
$$\psi'(y) \propto \int_{k_n-\Delta_k/2}^{k_n+\Delta_k/2} dk_x\, \psi(k_x, y)$$ → Altered predictions for measurements of y



Measure k_x≈k_0 →

Measure k_x≈k_1 →

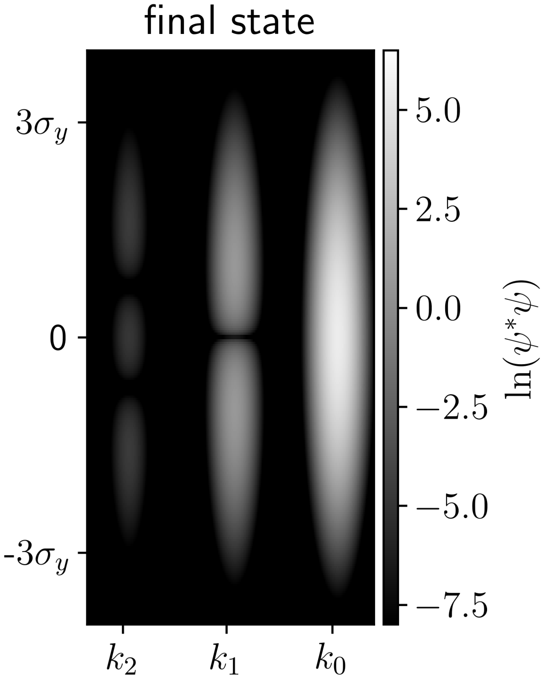
Measure k_x≈k_2 →



Entanglement between y and k_x

- Measuring *y* changes our predictions for k_x measurements

$$\psi'(k_x) \propto \int_{y_0-\Delta_y/2}^{y_0+\Delta_y/2} dy\, \psi(k_x, y)$$



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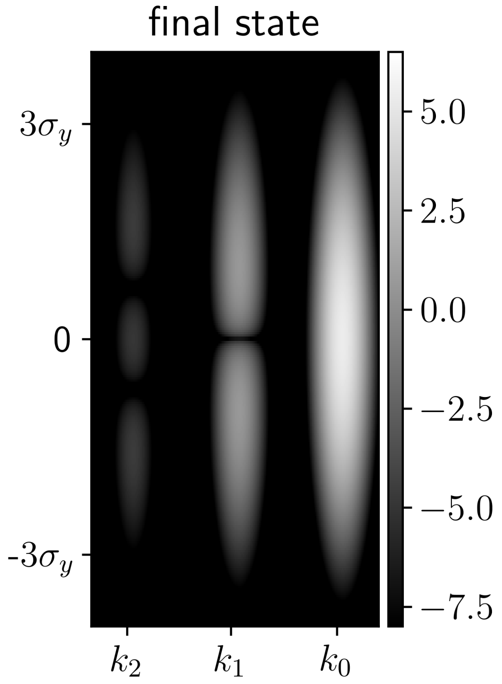


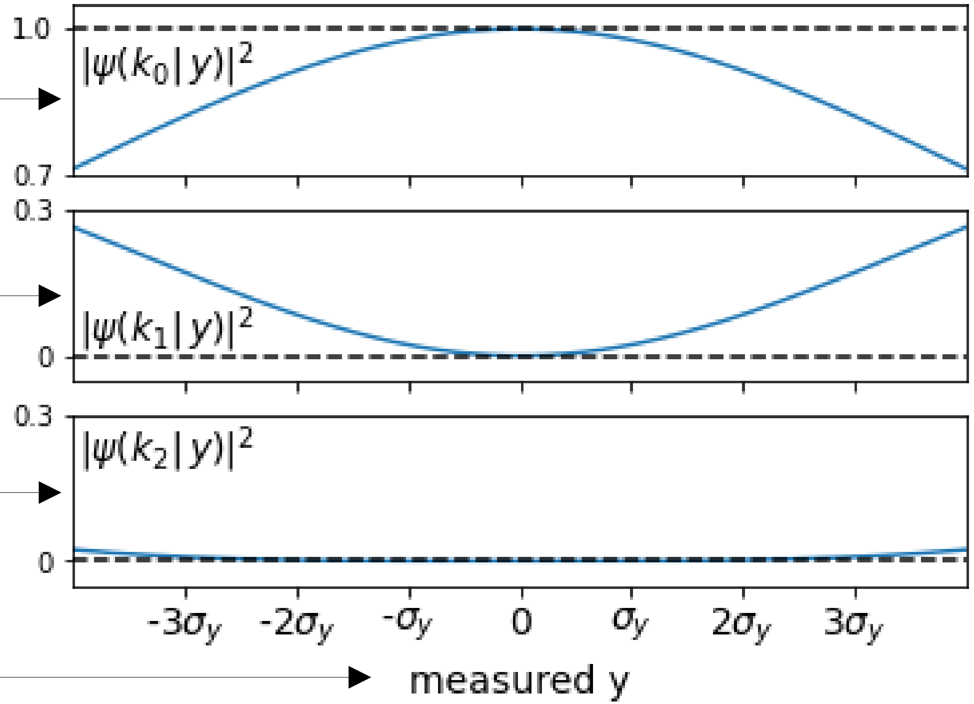
Altered predictions for measurements of k_x

k_0 probs

k_1 probs

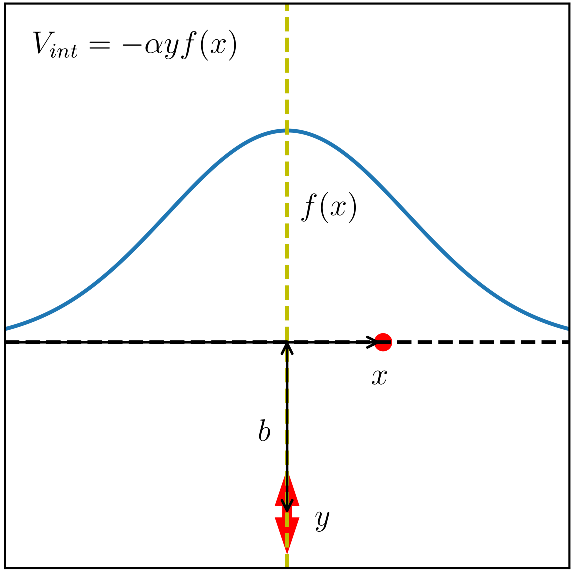
k_2 probs

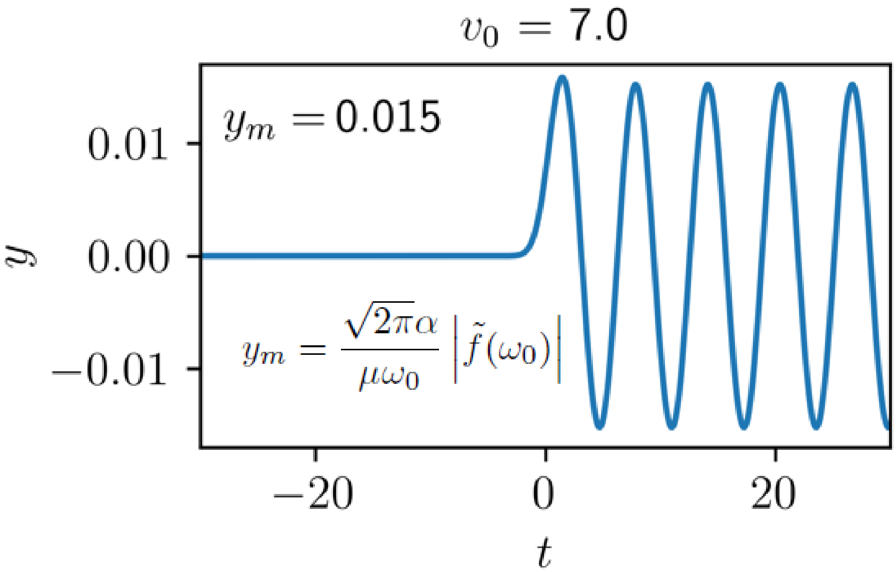
Measure y first



Conclusion

- In classical scattering, the external beam and target oscillator dynamics are correlated – but no one is impressed





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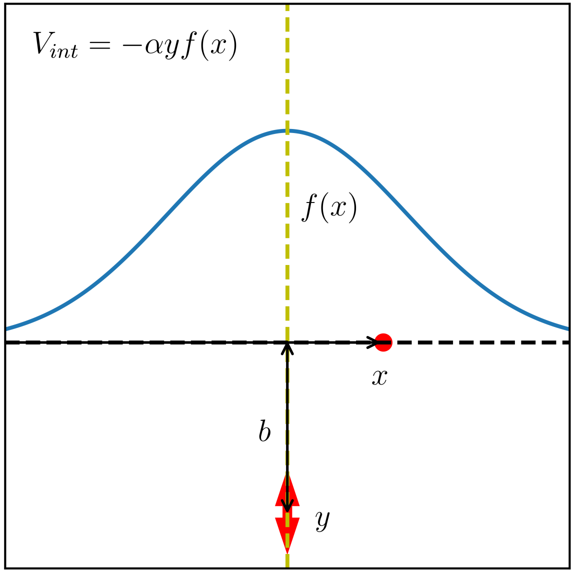
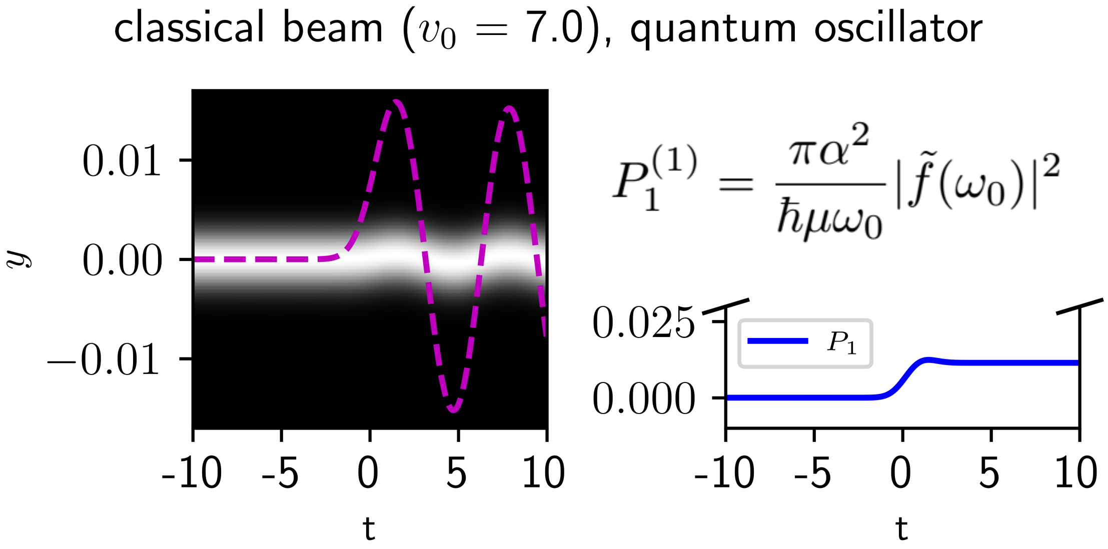
- In quantum textbook treatments, entanglement in scattering is obscured by using a classical beam that is not a probe





Conclusion

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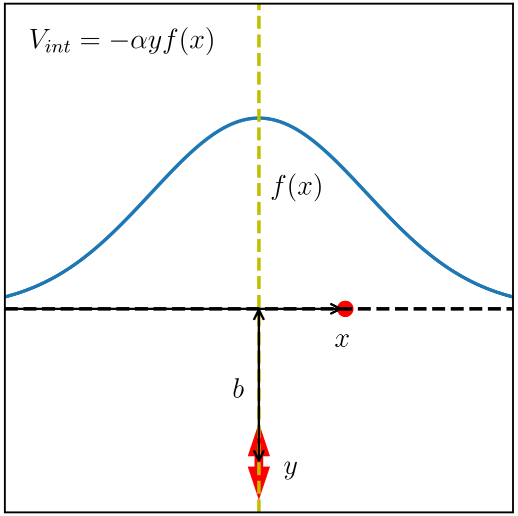
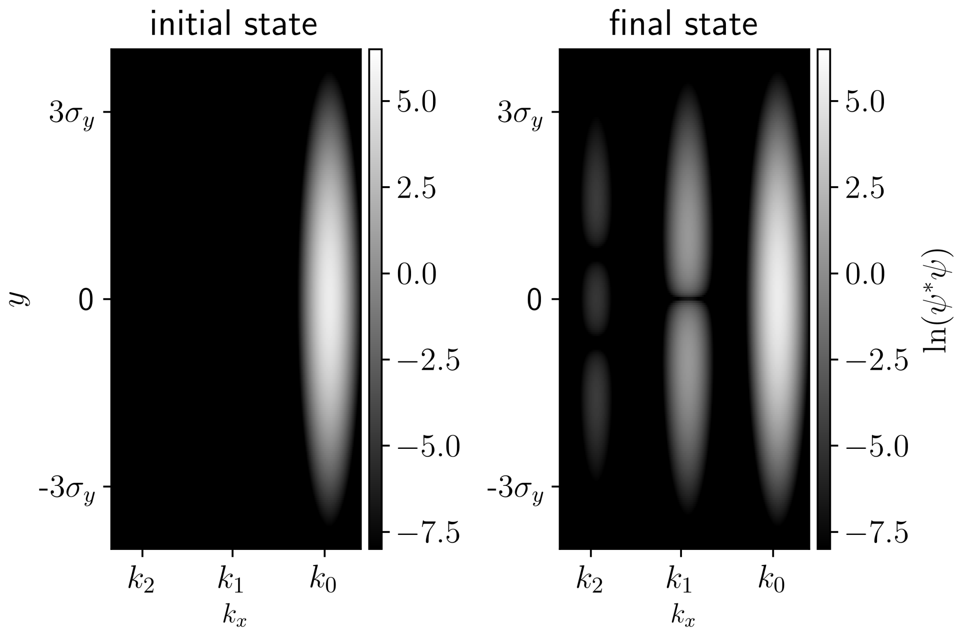
- In quantum textbook treatments, entanglement in scattering is obscured by using a classical beam that is not a probe
- Picturing the wavefunction density $\Psi^*\Psi$ in k_x and y can help us to picture scattering entanglement simply

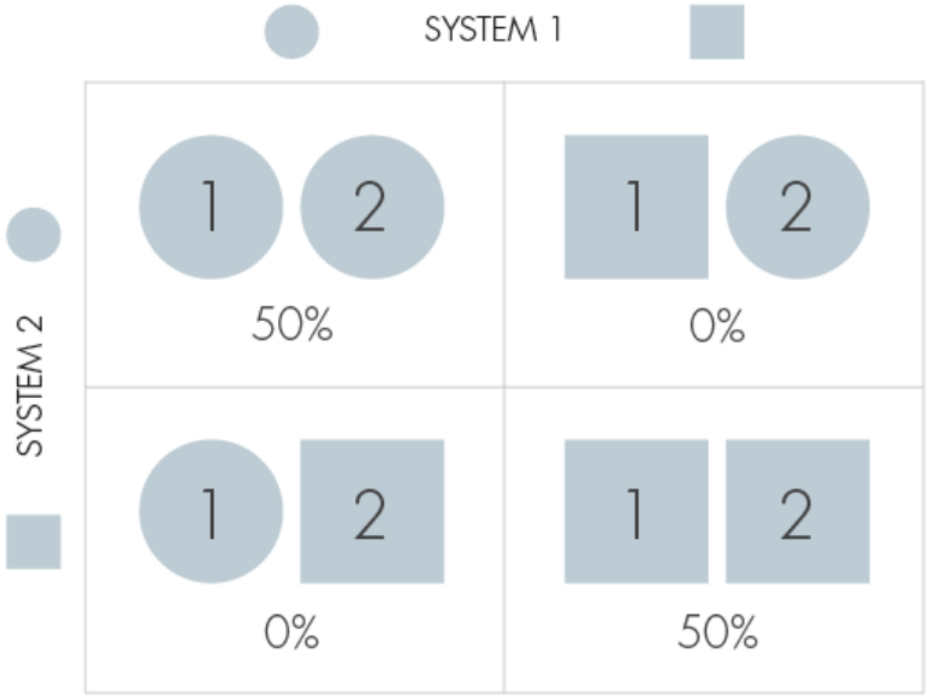




“Classical entanglement”

- Even in classical mechanics, there are plenty of cases where measuring a subsystem yields information about parts of the global system that are spatially separated from that subsystem

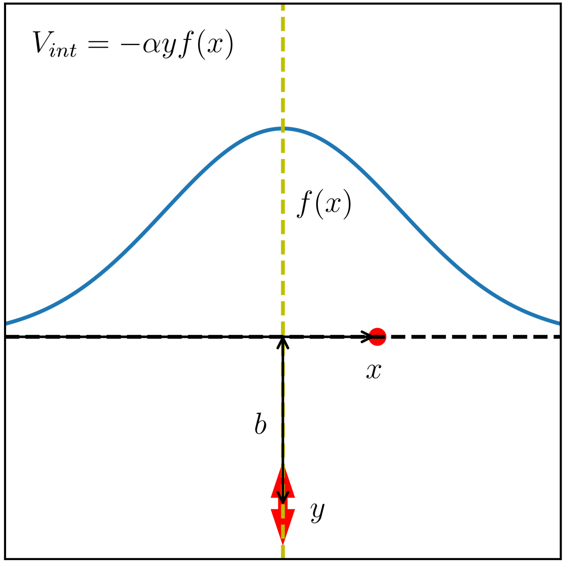
“Entanglement Made Simple,”
by Frank Wilczek (*Quanta*, 2016;
illustration by Olena Shmahalo)



Classical Examples

- It is straightforward to calculate the work done on a dipole oscillator constrained to oscillate along y

$$f(x) = \frac{b}{(x^2+b^2)^{-3/2}} \longrightarrow W = \frac{2\alpha^2\omega_0^2}{\mu v^4} K_1^2\left(\frac{b\,\omega_0}{v}\right)$$



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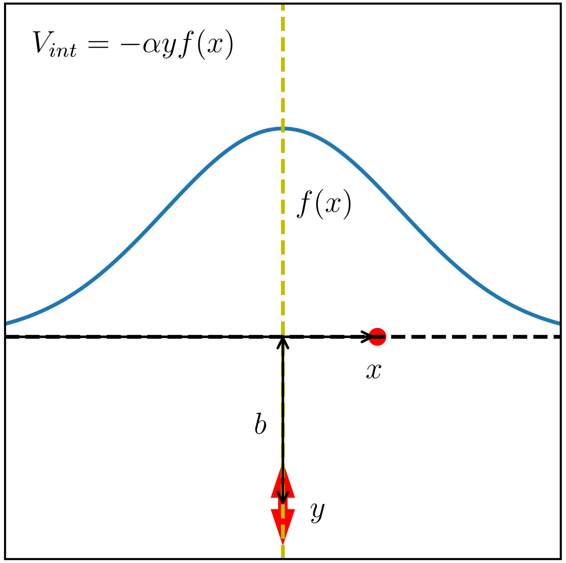
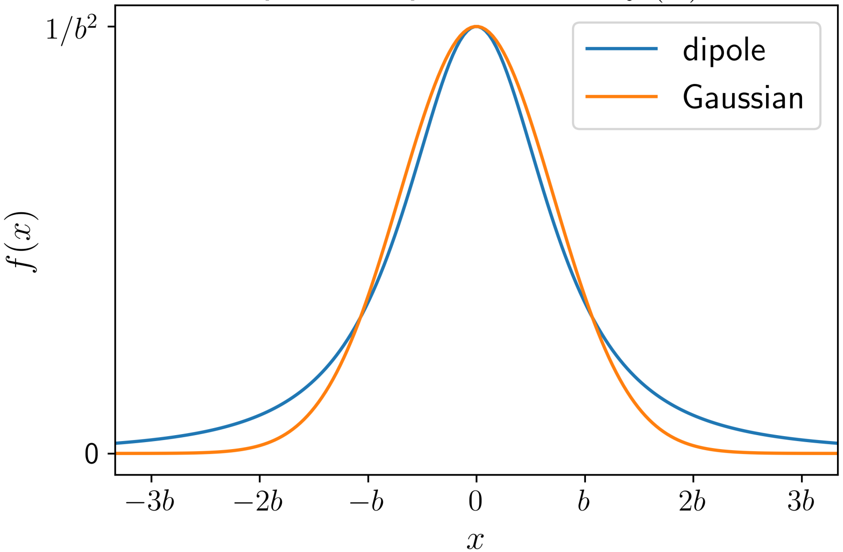
- Approximating this potential with a Gaussian makes the problem more tractable as homework

$$f(x) = b^{-2}e^{-x^2/b^2} \longrightarrow W = \frac{\pi\alpha^2}{2\mu v^2 b^2}e^{-b^2\omega_0^2/2v^2}$$





When Partly and Fully QM Approaches Align

- If we integrate the full Schrodinger equation over x, we can see how the interaction term complicates our picture

$$i\hbar\frac{\partial}{\partial t}\psi(y) = -\frac{\hbar^2}{2\mu}\frac{\partial^2}{\partial y^2}\psi(y) + \frac{1}{2}\mu\omega_o^2 y^2\psi(y) - \int dx\,\alpha\,y\,f(x)\,\psi(x,y)$$

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- If we compare this with the proposed dynamics of the reduced Schrodinger equation, we find that this works if

$$\int dx\, f(x)\, \psi(x,y) \approx f(vt) \int dx\, \psi(x,y)$$